Mathematical Reviews

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NUMBER THEORY

Fitting, F. Die Konstruktion magischer Quadrate von gerader Zellenzahl. Deutsche Math. **5**, 125–138 (1940). [MF 2814]

Each of the numbers $0, 1, \cdots, 4m^2-1$ can be expressed uniquely in the form (1) $\alpha_1+2\alpha_2+2\cdot 2\beta_1+2\cdot 2\cdot m\beta_2$, where $\alpha_i=0, 1, \beta_i=0, 1, \cdots, m-1$. (There are five other similar expressions obtained by using the factors $2\cdot 2\cdot m\cdot m$ of $4m^2$ in different orders.) If a magic square is constructed of the numbers $0, 1, \cdots, 4m^2-1$ and each number replaced by the corresponding value of α_1 (or of $\alpha_2, \beta_1, \beta_2$) in the expansion (1), the resulting square may or may not be magic. The author considers the construction of magic squares for which each of the four such "component" squares are magic. This is done by the use of certain square arrangements of order m, and empirical rules are given for constructing these arrangements. Each such arrangement gives rise to a large number of magic squares. *R. J. Walker* (Princeton, N. J.).

Narasimhamurti, V. On Demlo-numbers. Math. Student **8**, 34–37 (1940). [MF 2435]

The author proves a theorem about the number of integers having certain digital properties. *D. H. Lehmer.*

Pillai, S. S. On normal numbers. Proc. Indian Acad. Sci., Sect. A. **12**, 179–184 (1940). [MF 2948]

This paper is an addition to an earlier paper [Proc. Indian Acad. Sci., Sect. A. **10**, 13–15 (1939); these Rev. **1**, 4] in which some of the proofs were inadequate. The proof of Champernowne's theorem, that .123456789101112 $\cdots$ is normal, is corrected and rewritten and it is shown that the usual definition of a number being normal can be replaced by the simpler definition used in the first paper. The author states that the argument can be elaborated to complete the proofs of the other theorems of his earlier paper.

H. S. Zuckerman (Seattle, Wash.).

Vijayaraghavan, T. On decimals of irrational numbers. Proc. Indian Acad. Sci., Sect. A. **12**, 20 (1940). [MF 2749]

The paper contains a very short and simple proof of the theorem: Let $a_0.a_1a_2\cdots$ represent a number θ in the scale of m, $\theta_n=.a_na_{n+1}\cdots$ $(n=1, 2, 3, \cdots)$, and L be the set of limit points of $\theta_1, \theta_2, \cdots$. If θ is irrational, then L contains an infinity of points. A corollary is given: Let $\theta=\sqrt[n]{m}$ be irrational. Then the set of limit points of the fractional parts of the powers of θ contains an infinity of points.

The author does not point out the place of his problem within the framework of diophantine approximations. Hardy and Littlewood [Acta Math. **37**, 183 (1914)] say: "Perhaps the most interesting special sequence falling under the general type $(f(n)\theta)$ $[a=[a]+(a)]$ is that in which $f(n)=a^n$, where a is a positive integer. When θ is expressed as a decimal in the scale of a, the effect of multiplication by a is merely to displace the digits." Compare also Weyl [Math. Ann. **77**, 349 (1916)] and Koksma [Diophantische Approximationen, Ergebnisse der Math., vol. 4, 1935, p. 116]. The articles referred to give the theorem of the paper, even in much stronger form, for all irrational θ except an unspecified set of θ's of measure zero ("almost all" type of theorem). Vijayaraghavan's paper gives a much weaker theorem, but which holds for all irrational θ. *A. J. Kempner.*

Vijayaraghavan, T. On the fractional parts of the powers of a number. I. J. London Math. Soc. **15**, 159–160 (1940). [MF 2526]

This paper contains a proof due to the author and an alternative (somewhat simpler) proof due to A. Weil for the following theorem: If $\theta>1$ is rational, then there are infinitely many points of accumulation of the fractional parts of θ^n, $n=1, 2, \cdots$. *W. Feller* (Providence, R. I.).

Besicovitch, A. S. On the linear independence of fractional powers of integers. J. London Math. Soc. **15**, 3–6 (1940). [MF 2576]

Let $a_i=b_ip_i$, $i=1, \cdots, s$, where the p_i are s different primes and the b_i positive integers not divisible by any of them. The author proves by an inductive argument that, if x_j are positive real roots of $x^{n_j}-a_j=0$, $j=1, \cdots, s$, and $P(x_1, \cdots, x_s)$ is a polynomial with rational coefficients and of degree not greater than n_j-1 with respect to x_j, then $P(x_1, \cdots, x_s)$ can vanish only if all its coefficients vanish.

W. Feller (Providence, R. I.).

✱**Glaisher, J. W. L. Number-divisor Tables.** British Association for the Advancement of Science, Mathematical Tables, vol. 8. Cambridge University Press, Cambridge, England, 1940. x+100 pp. 15 s.

These tables are concerned with the three most important functions of the theory of numbers, namely Euler's totient function $\phi(n)$, giving the number of numbers not greater than n and prime to n, the number $\nu(n)$ of all the divisors of n, and $\sigma(n)$ the sum of these divisors. The volume is in four parts. The main table (Table I) gives for each $n\leqq 10000$ its canonical factorization into powers of primes, and the corresponding values of $\phi(n)$, $\nu(n)$, $\sigma(n)$. This occupies 63 pages. The other tables are inverse tables. Table II gives all solutions x of $\phi(x)=m$ for each possible $m\leqq 2500$. Table III gives all solutions $x\leqq 10000$ of $\nu(x)=m$ for each possible value of m (the largest being 64). Those solutions x which are products of distinct primes are printed in heavy type to facilitate the calculation and inversion of series involving Möbius function $\mu(n)$. Table IV gives all solutions x of $\sigma(x)=m$ for each possible $m\leqq 10000$.

The tables were first planned by Glaisher in 1882, completed to $n=3000$ and set in type in 1884. Later Table I was extended to 10000. The corresponding extension was still to be made when Glaisher died in 1928. In publishing these tables the British Association Tables Committee has extended and revised the inverse tables. The main table was left in Glaisher's original form. The introduction contains

errata in the three tables of Sylvester, Carmichael and Dickson which cover small parts of the present volume. There one also finds values of the three sum functions

$$\sum_{k=1}^{n}\phi(k),\quad \sum_{k=1}^{n}\nu(k),\quad \sum_{k=1}^{n}\sigma(k)$$

compared with the corresponding approximating functions

$$3n^2/\pi^2,\quad n(\log n+2c-1),\quad n^2\pi^2/12$$

for $n=1000(1000)10000$. These are by-products of the checking processes applied to the main table. These tables will be of considerable value to the future investigation of the many problems involving these fundamental numerical functions. *D. H. Lehmer* (Berkeley, Calif.).

Baidaff, B. I. The divisibility of integers by integers ending in 1, 3, 7, 9. Bol. Mat. **13**, 233–235 (1940). (Spanish) [MF 3152]

Sarantopoulos, Spyridion. Quelques théorèmes sur les nombres entiers. Bull. Soc. Math. Grèce **20**, 85–100 (1940). [MF 2632]

The results of this paper are based on the following lemma: If $a^\lambda\equiv 1 \pmod{M}$ and λ is prime to $\phi(M)$, then $a\equiv 1 \pmod{M}$. Thus a typical theorem is: If M divides x^u-y^λ and $xs^\lambda-1$, and λ is prime to $\phi(M)$, then M divides ys^u-1 and $x-ys^{u-\lambda}$. There are several such theorems about numbers of the form $x^u\pm y^\lambda$; in certain cases the form of the prime factors of such a number is determined. Some of the proofs could be simplified considerably.

H. W. Brinkmann (Swarthmore, Pa.).

Bauer, M. Zur Theorie der identischen Kongruenzen. J. London Math. Soc. **15**, 82–84 (1940). [MF 2515]

Let

$$S'(m, n)=\sum_{i=1}^{\phi(m)} r_i^n,\quad 0<r<m;\ (r_i, m)=1.$$

Gupta proved that (1) $kS'(m, 2j+1)\equiv 0 \pmod{m^2}$, where k is a suitable chosen divisor of $2(m, N_j)$,

$$N_j=\prod_{q-1\mid 2j} q,\qquad q \text{ prime.}$$

The author proved that (1) follows readily from the following identical congruence [M. Bauer, Bull. Soc. Phys.-Math. Kazan (3) **3** (1928)]: let $r_1, r_2, \cdots, r_{\phi(m)}$ be the reduced system of residues (mod m) and p an odd prime; further $m=p^a m'$, $(p, m')=1$. Then we have

$$\prod_{i=1}^{\phi(m)}(x-r_i^n)\equiv(x^{(p-1)/\delta}-1)^{\phi(m)/(p-1)\delta} \pmod{p^a},$$

where $(n, p-1)=\delta$. *P. Erdös* (Princeton, N. J.).

Grün, Otto. Eine Kongruenz für Bernoullische Zahlen. Jber. Deutsch. Math. Verein. **50**, 111–112 (1940). [MF 3070]

The congruence mentioned in the title is a special case of a theorem of Voronoi (1889) [see Uspensky and Heaslet, Elementary Number Theory, New York and London, 1939, p. 261]. Substituting this congruence into Kummer's celebrated criterion for Fermat's last theorem, the author obtains the equivalent criterion:

$$\sum_{n=1}^{l-1}\left[\frac{kn}{l}\right]\frac{t^n}{n}\equiv 0 \pmod{l}.$$

D. H. Lehmer (Berkeley, Calif.).

Pillai, S. S. On a linear Diophantine equation. Proc. Indian Acad. Sci., Sect. A. **12**, 199–201 (1940). [MF 2949]

The Diophantine equation $a_1x_1+\cdots+a_nx_n=1$, where $a_1, a_2, \cdots, a_n$ are integers with no common factor greater than 1, has solutions in integers $x_1, \cdots, x_n$. This note establishes an upper bound for $F=F(a_1, a_2, \cdots, a_n)$ defined as the least value (if any) of $|x_1|+\cdots+|x_n|$.

R. D. James (Saskatoon, Sask.).

Venkatarama Ayyar, M. and Bhimasena Rao, M. Types of solutions of $x^3+y^3+z^3=1$ in integers (I). J. Indian Math. Soc. (N.S.) **4**, 47–70 (1940). [MF 2691]

The authors begin by deriving a solution of the equation $x^3+y^3=z^3+w^3$ involving four parameters a, b, l, m. Here x, y, z, w are quadratic forms in a, b with coefficients that are functions of l, m. By specializing l, m they thus obtain various types of solutions. For example, if we use (L, M, N) as an abbreviation for $La^2+2Mab+Nb^2$, one solution is $(-14, -6, -54)^3+(14, -6, 54)^3=(7, -24, -27)^3+(-7, -24, 27)^3$; this particular one can be transformed into an identity due to Ramanujan. To solve the equation mentioned in the title, it is sufficient to have one of the four quadratic forms represent unity, either directly or after a common factor has been removed. Thus the illustration given above can be transformed into $(-12, 4, -8)^3+(10, 0, 8)^3=(1, 9, -4)^3+(-9, 7, 4)^3$ and $a=1$, $b=0$ gives us the solution $-12^3+10^3+9^3=1$ of the proposed equation. Many numerical solutions are given, among them a number that had been found by Ramanujan.

H. W. Brinkmann (Swarthmore, Pa.).

Reichardt, Hans. Über die Diophantische Gleichung $ax^4+bx^2y^2+cy^4=ez^2$. Math. Ann. **117**, 235–276 (1940). [MF 2155]

The solutions of the Diophantine equation

$$(1)\qquad ax^4+bx^2y^2+cy^4=ez^2$$

form a module under a certain rule of combination. This rule can be obtained either from the addition theorem for the elliptic functions corresponding to the equation (1) or from the law for the multiplication of the classes of divisors in the function field defined by (1). Mordell has proved [Proc. Cambridge Philos. Soc. **21**, 179–192 (1922)] that this module has a finite basis, but his proof gives no method of determining such a basis or of deciding the solubility of (1).

The author proves that the following problems are essentially equivalent. (1) The decision of the solubility and the determination of one solution. (2) The determination of a basis for the solutions of a solvable equation. (3) The determination of all the equations which are birationally transformable into the equation

$$x^4+bx^2y^2+cy^4=z^2.$$

To solve one of these problems for a fixed equation of the form (1), it is sufficient to solve one of the other two problems for certain equations of the form (1), finite in number.

A. Brauer (Princeton, N. J.).

Noguera, Rodrigo. The Goldbach-Waring theorem. Bol. Mat. **13**, 224–228 (1940). (Spanish) [MF 3150]

Gupta, Hansraj. On the absolute weight of an integer. Proc. Indian Acad. Sci., Sect. A. **12**, 60–62 (1940). [MF 2752]

The problem discussed in this paper was first considered by Sambasiva Rao and Pillai [J. Indian Math. Soc. (2) **3**,

262–265 and 266–267 (1939); these Rev. 1, 135]. Any positive integer x can be represented uniquely in the form $x=x_1^k+x_2^k+\cdots+x_s^k$, where x_i^k is the largest kth power not exceeding $x-x_1^k-x_2^k-\cdots-x_{i-1}^k$, $i=1, \cdots, s-1$, and $x_s^k=x-x_1^k-\cdots-x_{s-1}^k$. Let $S_k(x)$ denote the number of kth powers required in the above representation. Let N_j be any integer such that $S_k(N_j)\geqq S_k(x)$ for all $x\leqq N_j$ and let N_{j+1} be the least integer greater than N_j for which $S_k(N_{j+1})=S_k(N_j)+1$. Then it is shown that

(A) $j=[\{\log\log N_{j+1}-\log\log r^k\}/\{\log k-\log(k-1)\}]$,

where r is some number such that $2\leqq r\leqq 2k$.

R. D. James (Saskatoon, Sask.).

Pillai, S. S. A note on Gupta's previous paper. Proc. Indian Acad. Sci., Sect. A. 12, 63–64 (1940). [MF 2753]

The author notes that the result of the previous paper by Gupta leads to

$$\max S_k(x)=\{\log\log x-\log k\}/\{\log k-\log(k-1)\}+M(2^k k^k)+O(k\log\log k),$$

where $M(n)=\max S_k(x)$ for $1\leqq x\leqq n$. He goes on to show that a modification of Gupta's method will prove that

$$\max S_k(x)=\{\log\log x-\log\log(2^{k-1}k^{k-1})\}/\{\log k-\log(k-1)\}+M(2^{k-1}k^{2k-1})+O(1).$$

R. D. James (Saskatoon, Sask.).

Pillai, S. S. Waring's problem with indices $\geqq n$. Proc. Indian Acad. Sci., Sect. A. 12, 41–45 (1940). [MF 2751]

Let $g_2(n)$ denote the least value of s such that the equation $N=u_1+u_2+\cdots+u_s$, where each u_i is of the form x^m with $m\geqq n$, has a solution for every positive integer N. Miss Haberzetle has proved [Duke Math. J. 5, 49–57 (1939)] that $g_2(n)=2^n+k-1$, where $k=[\log l/\log 2]$ and $l=[3^n/2^n]$, on the assumption that $n\geqq 9$ and $3^n-l2^n\leqq 2^n-k-3$. Her proof is based on that given by Vinogradov [Ann. of Math. (2) 36, 395–404 (1935)] for Waring's problem together with Dickson's method of ascent [Amer. J. Math. 58, 521–529 (1936)]. Pillai notes that the problem under consideration is just made for representation in the scale of j for certain values of j. For, if $x=c_0+c_1j+\cdots+c_ij^i$, then $xj^n=c_0j^n+c_1j^{n+1}+\cdots+c_ij^{n+i}$. With this method instead of Dickson's he proves that $g_2(n)=2^n+k-1$ for all $n\geqq 32$ and no other restrictions. *R. D. James* (Saskatoon, Sask.).

Pillai, S. S. On Waring's problem with powers of primes. Proc. Indian Acad. Sci., Sect. A. 12, 202–204 (1940). [MF 2950]

Let θ be determined so that p^θ divides k but $p^{\theta+1}$ does not divide k; let $\gamma=\theta+2$ for $p=2$ and k even, $\gamma=\theta+1$ otherwise. Let $p_1, p_2, \cdots, p_n$ denote primes for which p_i-1 divides k, and write $K=p_1^{\gamma_1}\cdots p_n^{\gamma_n}$. In Waring's problem with powers of primes it is necessary to have some upper bound for the least value of s such that the congruence $N\equiv x_1+\cdots+x_s \pmod K$ has a solution for every integer N. Here $x_1, \cdots, x_s$ can be 0, p^k or P^k, where p is any prime dividing K and P is the least prime which does not divide K. Such a bound is determined in this note. *R. D. James.*

Gupta, Hansraj. Waring's problem for powers of primes. II. J. Indian Math. Soc. (N.S.) 4, 71–79 (1940). [MF 2692]

[Part I appeared in J. Indian Math. Soc. (2) 3, 136–145 (1938).] Let n be any integer; write $3^n=2^nq+r$, where $0\leqq r<2^n$, and let $I=2^nq-2$. It is conjectured that for $n\geqq 7$ every integer is the sum of at most I nth powers of primes. Numerical evidence is put forward to support this conjecture when $7\leqq n\leqq 19$, a typical result being the following: every integer less than 41^7 is the sum of at most 143 seventh powers of primes. *R. D. James* (Saskatoon, Sask.).

Auluck, F. C. On Waring's problem for biquadrates. Proc. Indian Acad. Sci., Sect. A. 11, 437–450 (1940). [MF 2605]

It was first proved by Hardy and Littlewood that every integer exceeding a certain large number C could be represented as the sum of 19 integral fourth powers. In this paper the numerical values of the constants in the Gelbcke modification of the Hardy-Littlewood method are worked out and it is proved that $\log_{10}\log_{10} C\leqq 88.39$. The introductory remarks lead one to believe that it will be shown that $\log_{10}\log_{10} C\geqq 88.39$. This is not the case.

R. D. James (Saskatoon, Sask.).

Hua, Loo-Keng. Sur le problème de Waring relatif à un polynome du troisième degré. C. R. Acad. Sci. Paris 210, 650–652 (1940). [MF 3038]

Let $f(x)=a(x^3-x)/6+b(x^2-x)/2+cx+d$, $a>0$, $(a, b, c)=1$, a, b, c, d integers. Two theorems are stated: I. Every large integer is a sum of eight values $f(x)$, $x=0, 1, 2, \cdots$. II. Almost all positive integers are sums of four values (and 4 is best possible); except that 4 must be replaced by 7 if $f(x)\equiv 2(2a'+1)x^3+(2b'+1)x^2+2(2c'+1)x+d' \pmod{16}$, and by 5 if $f(x)$ is not of this form but $f(x)\equiv a''x^3+3b''x^2+3c''x+d'' \pmod 9$ with $b''^2\equiv a''(b''+a'') \pmod 3$. The second theorem requires a lemma analogous to one of Davenport [Acta Math. 71, 123–143 (1939); these Rev. 1, 5] and a result by the author [the same C. R. 210, 520–523 (1940); these Rev. 2, 40], and the following result which is proved here: the number of solutions of $f(x_1)+f(x_2)+f(x_3)+f(x_4)\equiv n \pmod p$ is $p^3+O(p^2)$. *G. Pall* (Princeton, N. J.).

Hall, Newman A. The number of representations function for binary quadratic forms. Amer. J. Math. 62, 589–598 (1940). [MF 2456]

Explicit formulae are obtained for the number of representations of an arbitrary integer in a positive binary quadratic form, for all cases in which there is a genus of one class. *G. Pall* (Princeton, N. J.).

Weyl, Hermann. Theory of reduction for arithmetical equivalence. Trans. Amer. Math. Soc. 48, 126–164 (1940). [MF 2510]

Minkowski's investigations in the geometry of numbers started from the problem of reduction for positive quadratic forms of n variables, but he found later some considerable difficulties in the application of his geometrical method to this problem and finally solved it directly by arithmetical ideas. The author sets forth a more geometrical theory of reduction which depends upon Minkowski's fundamental inequality (1) $S_1\cdots S_nV\leqq 2^n$ concerning a symmetric convex body in relationship to a lattice. In this inequality appear n vectors $v_1, \cdots, v_n$ of the lattice which generally do not constitute a basis. The relationship (1) is replaced by another inequality in which the vectors $v_1, \cdots, v_n$ form a basis of the lattice. It is remarked that this result had already been obtained independently by K. Mahler [Quart. J. Math. 9, 259–262 (1938)]. The new inequality gives rather strong estimates for certain quantities in the theory of reduction. In the second chapter the convex body is specialized to the case of an ellipsoid. The

results of the first chapter lead to a finite number of conditions which define a reduced positive quadratic form. This gives Minkowski's theorem that the domain of reduced positive quadratic forms is a convex polyhedron Z with a finite number of faces. By any unimodular substitution S of the variables, Z is carried into an equivalent cell Z_S, and these equivalent cells cover without gaps and overlappings the domain G of all positive quadratic forms. Minkowski had proved that Z borders on not more than a finite number of equivalent cells Z_S. The author obtains the sharper result that into any compact subset of G penetrate only a finite number of cells Z_S.

The whole theory is extended to lattices and forms in which complex numbers or quaternions take the place of real numbers, under the essential conditions that every ideal in the corresponding ring of integers is a principal ideal.

C. L. Siegel (Princeton, N. J.).

Linnik, U. V. On certain results relating to positive ternary quadratic forms. Rec. Math. [Mat. Sbornik] N.S. **5** (47), 453–471 (1939). (English. Russian summary) [MF 1337]

The author studies the problem of representation of integers m by a positive ternary quadratic form F. He associates a suitably chosen algebra of generalized quaternions with every such form F and expresses the diophantine equation $F=m$ as a quaternion equation. In the process of solution of this latter equation the author employs the automorphs of F expressed in terms of generalized quaternions and investigates in detail the related properties of binary quadratic forms of discriminant $-m$. The principal results are proven under certain restrictions on m for forms with the invariants $\Omega=p^a$, $\Delta=1$, where p is an odd prime, and enable one to study representation of integers by individual forms with such invariants, in a genus containing more than one class [cf., for similar results, B. W. Jones and G. Pall, Acta Math. **70**, 165–191 (1939)]. In the addendum the author states that by a combination of the algebraic results of this paper and an analytic method similar to Viggo Brun's sieve he has proven that every positive ternary quadratic form represents every sufficiently large integer satisfying its generic conditions and certain supplementary congruence conditions with respect to an arbitrary fixed system of a sufficiently large system of moduli. Such a theorem would be an important extension to ternary forms of the results of Tartakowsky [Bull. Acad. Sci. URSS [Izvestia Akad. Nauk SSSR] (7) **1929**, 111–122, 165–196] for quadratic forms in four or more variables.

A. E. Ross (St. Louis, Mo.).

Linnik, J. On the representation of large numbers by positive ternary quadratic forms. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 575–576 (1939). [MF 2080]

In this note the author gives the following results: Let f be a positive ternary quadratic form of invariants $\Omega=p$ (p an odd prime), $\Delta=1$ and such that $(f|p)=(-1|p)$, $f\neq 8b+7$. Then f represents all the sufficiently large integers m not divisible by p and satisfying the last two conditions for f. Next let F be the reciprocal of the form f above. Then all sufficiently large integers m which are prime to p and for which $F\equiv m$ (mod 8) is solvable, are represented by F. For the method of proof one is referred to a former note [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 211–212 (1939)]. These results are special cases of a general theorem stated at the conclusion of another paper [cf. the preceding review].

A. E. Ross (St. Louis, Mo.).

Linnik, U. V. Some new theorems on the representation of large numbers by separate positive ternary quadratic forms. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 211–212 (1939).

In this paper the author gives some results concerning the representation of large integers by positive ternary quadratic forms of order $[\Omega, 1]$ and generic invariants $f\not\equiv 7$ (mod 8) and $(f|\omega)=(-1|\omega)$ for all $\omega|\Omega$. Following some special representation theorems, the author observes (Th. III) that one can distribute all integers compatible with the above generic invariants into an infinite sequence of arithmetic progressions $\mathfrak{A}_i$ so that each form of the genus in question should represent all sufficiently large integers m ($m>c_i$) in each of these progressions. Here the lower bound c_i for such m depends on $\mathfrak{A}_i$. In a later note [see the preceding review] a stronger result is given, subject to the restriction that Ω be an odd prime. There is a brief outline of the methods employed. Complete proofs are promised in a forthcoming paper.

A. E. Ross (St. Louis, Mo.).

Pall, Gordon. On the rational automorphs of $x_1^2+x_2^2+x_3^2$. Ann. of Math. (2) **41**, 754–766 (1940). [MF 3018]

The automorphs are represented explicitly each as a matric function $A(t)$ of a quaternion t. This matric function has the multiplicative property $A(t)A(u)=A(tu)$. This leads to the factorization of automorphs of denominator $m=m_1m_2$ as a product of automorphs of denominators m_1 and m_2. Application is made to transforming integral solutions of $x_1^2+x_2^2+x_3^2=n$ into other integral solutions. Throughout the paper results are obtained with elegance and precision and with many details that cannot be indicated here.

M. Hall (New Haven, Conn.).

Pall, Gordon. On the arithmetic of quaternions. Trans. Amer. Math. Soc. **47**, 487–500 (1940). [MF 2163]

In this paper the author studies certain divisibility properties of the Lipschitz integral quaternions (that is, those with integral coordinates) and applies his results to the study of the connection between the congruence $v_1^2+v_2^2+v_3^2\equiv 0$ (mod m) and the equation $t_0^2+t_1^2+t_2^2+t_3^2=m$, and to the derivation of known relations concerning binary quadratic class numbers and representation by the sum of three squares.

A. E. Ross (St. Louis, Mo.).

Braun, Hel. Geschlechter quadratischer Formen. J. Reine Angew. Math. **182**, 32–49 (1940). [MF 2377]

In this paper the author gives the necessary and sufficient conditions, involving Gauss sums, for the existence of quadratic forms with integral coefficients possessing a given set of generic invariants. Here forms f are represented by their symmetric matrices $\mathfrak{S}$. Two symmetric matrices $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are said to be equivalent if there exist integral matrices $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{A}'\mathfrak{S}_1\mathfrak{A}=\mathfrak{S}_2$ and $\mathfrak{B}'\mathfrak{S}_2\mathfrak{B}=\mathfrak{S}_1$. Should $\mathfrak{A}'\mathfrak{S}_1\mathfrak{A}\equiv\mathfrak{S}_2$ and $\mathfrak{B}'\mathfrak{S}_2\mathfrak{B}\equiv\mathfrak{S}_1$ (mod q), then $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equivalent modulo q. Matrices equivalent modulo any integer q and moreover equivalent in the field of real numbers form a genus. Two matrices $\mathfrak{S}_1$ and $\mathfrak{S}_2$ of order m belong to the same genus if and only if (i) they have the same index μ, (ii) the same absolute value D of the determinant and (iii) if they belong to the same class modulo q_0, where $q_0=8D^3$. [H. Minkowski, Gesammelte Abhandlungen, 1911, vol. 1, p. 158; C. L. Siegel, Ann. of Math. (2) **36**, 527–606 (1935); in particular, p. 548.] Employing the system (i)–(iii) of generic invariants, the author proves that the necessary and sufficient conditions for the existence of the

genus with assigned invariants are: (1) $|\mathfrak{S}| = x^2(-1)^{m-\nu}D$ (mod q_0) has a solution x prime to q_0,

$$(2) \qquad \sum_{\mathfrak{x}(q_0)} e^{(2\pi i/q_0)\mathfrak{x}'\mathfrak{S}\mathfrak{x}} = e^{(\pi i/4)(2\nu-m)}(2q_0)^{m/2}D^{\frac{1}{2}}.$$

Here the elements $x_1, \cdots, x_n$ of the column vector $\mathfrak{x}$ in the Gauss sum in (2) run independently through a complete residue system modulo q_0. In the conclusion of the paper it is shown that these results yield a more direct proof of two important lemmas by Siegel [l.c., pp. 550 and 559].

A. E. Ross (St. Louis, Mo.).

Behrens, Ernst-August. Bestimmung der Stufe für die aus binären Thetareihen erzeugten Modulformen. Math. Z. 46, 350–374 (1940). [MF 2404]

The theta series $\vartheta(\tau; \rho, \mathfrak{a}, Q\sqrt{D})$ were discussed by E. Hecke [Math. Ann. 97, 210–242 (1926)], who showed that they remain invariant under the principal congruence subgroup $\Gamma(Q|D|)$ of Stufe $Q|D|$. In this paper it is shown that, for the case $Q=1$ and a fixed D, there is no linear combination of the theta series which remains invariant under a principal congruence subgroup $\Gamma(R)$ with $|D|=pR$ and p a prime. From this it follows that the functions $\vartheta(\tau; \rho, \mathfrak{a}, \sqrt{D})$ and their linear combinations are modular forms of the exact Stufe D, the discriminant of the quadratic field used in defining them. *H. S. Zuckerman* (Seattle, Wash.).

Sugawara, Masao. Über eine allgemeine Theorie der Fuchsschen Gruppen und Theta-Reihen. Ann. of Math. (2) 41, 488–494 (1940). [MF 2549]

In a recent paper [Math. Ann. 116, 617–657 (1939); these Rev. 1, 203] Siegel generalized the modular functions; Sugawara's paper is devoted to the analogous generalization of the automorphic functions. Denote by U_i $(i=1, \cdots, 4)$, P, Q, Z matrices of order n, by $\bar{U}_i$, $\bar{P}$, $\cdots$ their complex conjugate matrices, and let E be the identical matrix. Then

$$T=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \quad S=\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$$

are matrices of order $2n$. Denote by Γ the group of the matrices U satisfying the equations $U'TU=T$, $U'S\bar{U}=S$, and by $\mathfrak{A}$ the set of all symmetric matrices Z satisfying the inequality $E>Z'\bar{Z}$ (which means that the Hermitian form corresponding to the matrix $[E-Z'Z]$ is definite positive). For every matrix U of Γ the author considers the transformation

$$Z_1=\sigma(Z)=(U_1Z+U_2)(U_3Z+U_4)^{-1}$$

and proves that these transformations transform $\mathfrak{A}$ into itself and form a group. A subgroup G is called a general Fuchsian group if it contains no infinitesimal transformation. The author proves that such a group is properly discontinuous and that the corresponding series

$$\Theta(Z)=\sum |U_3Z+U_4|^{-k(n+1)}, \qquad k\geqq 2,$$

converges in $\mathfrak{A}$ and defines there an analytical function which satisfies the equation

$$\Theta(\sigma(Z))=|U_3Z+U_4|^{k(n+1)}\Theta(Z),$$

if $\sigma(Z)$ is the most general transformation of G. Finally, the author proves that Siegel's modular group can be considered as a particular case of his Fuchsian groups G.

It may perhaps be interesting to compare these important results with the properties of the discontinuous groups of movements in a Riemannian geometry and with Mirberg's theorems on the groups of birational transformations.

G. Fubini (Princeton, N. J.).

***Weyl, Hermann. Algebraic Theory of Numbers.** Annals of Mathematics Studies, no. 1. Princeton University Press, Princeton, N. J., 1940. viii+223 pp. $2.35.

This book consists of the notes of a course given by the author at the Institute for Advanced Study during the year 1938–1939. The first chapter gives a short but rather complete description of the algebraic properties of a field κ which is of finite degree over a given ground field k. It includes, for instance, a study of the discriminant, of the question of separability, of the Galois group, etc. In Chapter II, the theory of divisibility is developed. Here, the author prefers Kronecker's theory, presented in a modernized axiomatic form, to Dedekind's theory of ideals. Let k be a field, and let $[k]$ be a subring containing 1 whose quotient field is k; the elements of k will be denoted as integers. Besides the integers, we have a second class of objects $\mathfrak{a}, \mathfrak{b}, \cdots$, the divisors. The basic relation is that of divisibility of an integer by a divisor. These concepts are introduced by a set of axioms. For instance, the first two axioms require that the integers divisible by a fixed divisor form an ideal and that the divisor is characterized uniquely by this ideal. Further, we have a unit divisor $\mathfrak{e}$ which divides every integer. The divisibility of a divisor $\mathfrak{a}$ by a divisor $\mathfrak{b}$ is defined to mean that every integer divisible by $\mathfrak{a}$ is divisible by $\mathfrak{b}$. A further group of axioms deals with the multiplication of divisors and another one with the multiplication of a divisor by an integer. Finally, three axioms of more decisive nature are needed: (1) To every divisor $\mathfrak{a}$, there exists a divisor $\mathfrak{a}'$ such that $\mathfrak{a}\mathfrak{a}'$ is principal, that is, of the form $a\mathfrak{e}$, a in $[k]$. (2) For every $\mathfrak{a}$, there exists a natural number h such that $\mathfrak{a}$ allows no factorization into more than h factors which are all different from $\mathfrak{e}$. If h can here be taken as 1, then $\mathfrak{a}$ is said to be a prime divisor. (3) If a product of divisors is divisible by a prime divisor $\mathfrak{p}$, then one of the factors is divisible by $\mathfrak{p}$. From these axioms it follows at once that the decomposition of a divisor into prime factors is unique; we can introduce the greatest common divisor as well of integers as of divisors, and then the content (Inhalt) of a rational function of indeterminates $x, y, \cdots$ with coefficients in k can be defined. We now suppose we are given a ground field k in which integers and divisors are defined in accordance with the axioms; for instance, the field of rational numbers can be taken for k. Let κ be an extension field of k of finite degree. The aim is to extend the notions "integer" and "divisor" without destroying the validity of the axioms. This is accomplished by the use of Kronecker's fundamental idea. Instead of operating with divisors, rational functions of indeterminates are used, and the divisors appear as the contents of these functions. An element of the field $k(x, y, \cdots)$ of rational functions is integral if its content is integral. An element of $\kappa(x, y, \cdots)$ is integral if it is integrally dependent on integral elements of $k(x, y, \cdots)$. We further have the definition: Any finite set of integers $(\alpha_1, \cdots, \alpha_r)$ of κ which do not all vanish determines a divisor $\mathfrak{A}$ belonging to κ; an integer α is divisible by $\mathfrak{A}$ if and only if $\alpha/(\alpha_1x_1+\cdots+\alpha_rx_r)$ is an integral element of $\kappa(x_1, \cdots, x_r)$. There is no difficulty in defining the multiplication of these divisors $\mathfrak{A}$. It can be shown that all the axioms hold for κ. The field k is said to be a Dedekind field when it has the following property: Whenever the integers $a_1, \cdots, a_r$ of k are without common divisor $\neq \mathfrak{e}$, then 1 can be written as a multiple sum of $a_1, \cdots, a_r$ in $[k]$. The theory is completed by the proof of the theorem that, if k is a Dedekind field, so is the extension κ of finite degree. A highly interesting comparison between

Kronecker's and Dedekind's viewpoints concludes the chapter. Some readers may perhaps have the impression that the author had a slight prejudice against the theory of ideals when he wrote these pages. Apparently, Weyl himself thought so when he added the amendment at the end of the book.

The third chapter discusses "local primadic analysis." The ground field k is taken as a Dedekind field; the decomposition of a prime ideal $\mathfrak{p}$ of k in an extension field κ of degree n is studied. The reader is prepared for Hensel's theory by a study of the quadratic case and, more generally, by that of Kummer's case where κ has an integral basis $1, \vartheta, \cdots, \vartheta^{n-1}$ consisting of the powers of one element. Then p-adic numbers are introduced, and their application to the problem in question is given. There follows an exposition of Hilbert's theory of Galois fields, and the Artin symbol is defined. As an example, the cyclotomic fields are studied in all detail. In the last chapter, the fields are assumed to be algebraic number fields. The first sections complete the elementary theory, dealing with the integral basis of an ideal, the connection between norm and number of residue classes, Euler's function and Fermat's theorem. The infinite prime ideals are introduced, the notion of "prime spots" appears, and the analogy to the points of a Riemann surface is discussed. Next follows the proof of Minkowski's theorem on linear forms, based on Minkowski's geometric principle, and the applications of Minkowski's theorem, the finiteness of the class number, the existence of ramification spots in the case of the rational ground field. Further, the theory of units is developed in the form in which it is needed in class field theory. The following sections deal with the asymptotic distribution of the ideals over the classes; the ζ-function and the L-series are introduced, and the connection with the question of the equidistribution of the prime ideals over the classes is discussed. For the case of rational primes, Dirichlet's theorem on the primes in arithmetic progressions is proved. Then the case of a quadratic field is studied, and the explicit class number formulae are given. The book is concluded with a discussion of the norm residue symbol, first in the quadratic case, and then, following Chevalley, in the general case; here only part of the proofs are given. In this connection, the aim of the class field theory is described, and some of its most important results are indicated.

This description shows the amazing amount of material covered in the book; however, every important new concept is carefully prepared, and the reader actually sees the reasons which lead to the introduction of the concept. The essential steps in the proofs are shown, and the significance of the results is discussed. The reader is not just taken on a sight seeing tour, similar to many others of the same kind. Actually, less frequented parts are visited, surprising views are obtained, and a deep insight into the algebraic theory of numbers is gained. *R. Brauer* (Toronto, Ont.).

Rédei, László. Über den Euklidschen Algorithmus in reell quadratischen Zahlkörpern. Mat. Fiz. Lapok 47, 78–90 (1940). (Hungarian. German summary) [MF 2655]

Let m be a composite number. It is well known that, if in the field $R(\sqrt{m})$ the Euclidean algorithm exists, then $m=pq$ (p, q primes). The author proves, by using a formula of Hasse, that $m=3q$. Previously Schuster proved [Monatsh. Math. Phys. **47**, 117–127 (1938)] that the case $m=3q$ is impossible except for $m=21$, 33, 57. Thus the results of Rédei and Schuster imply that if m is composite the Euclidean algorithm can exist only for $m=21$, 33 and 57. Some time ago Heilbronn proved that the Euclidean algorithm can exist only for a finite number of m [H. Heilbronn, Proc. Cambridge Philos. Soc. **34**, 521–526 (1938)]. *P. Erdös* (Philadelphia, Pa.).

Lednew, N. A. Über die Einheiten der relativ zyklischen algebraischen Zahlenfelder. Rec. Math. [Mat. Sbornik] N.S. **6** (48), 227–261 (1939). (Russian. German summary) [MF 1356]

The author studies in detail the structure of the group of units of an algebraic field. He generalizes, to the case of a relative cyclic extension of any degree, the theorem of Hilbert [Gesammelte Abhandlungen, Band 1, p. 150; Jber. Deutsch. Math. Verein. **4**, 273 (1897)] on the existence of the system of relative fundamental units for a cyclic extension of prime degree. The author's results contain the important special case of relative cyclic extension in the theorem of Herbrand [C. R. Acad. Sci. Paris **191**, 1282 (1930); cf. E. Artin, J. Reine Angew. Math. **167**, 153–157 (1931)]. The author proves also that, should k be a real algebraic field (of finite degree) whose conjugate fields are all real, should p be a prime not dividing the discriminant of k and ζ a primitive root of unity of degree p^m, then every fundamental system of units of $k(\zeta+\zeta^{-1})$ is a fundamental system for $k(\zeta)$. This is a two-fold generalization of a similar theorem for $m=1$ and k a field of rational numbers [Hilbert, ibid., pp. 204 and 336, respectively]. *A. E. Ross.*

Chevalley, C. La théorie du corps de classes. Ann. of Math. (2) **41**, 394–418 (1940). [MF 1821]

The words "class field theory" have by now obtained a very general meaning. They are used whenever an extension field of an algebraic number field is determined by the original field. Class field theory in the original sense dealt with all finite abelian extension fields of a given field and their connections with the class groups in the original field. The author deals with all finite or infinite abelian extension fields and their connections with the original field; the words "class group," however, hardly occur in the whole paper. Another striking difference of this paper from earlier work on class field theory [e.g., the author's thesis, J. Fac. Sci. Univ. Tokyo **2**, 365–476 (1933)] is the exclusion of analytical methods. The theory of the ζ-functions which for so long a time seemed necessary can now be omitted. On the other hand, topological methods play an important part in this new presentation of class field theory. [The use of topological methods in this connection was also suggested by D. van Dantzig, Ann. École Norm. (3) **53**, 275–307 (1936), especially 281.] A topology is introduced into two groups: (1) The Galois group G of the maximum abelian extension field A of the given field K (that is, the field composed of all abelian extension fields of K). This group is made into a topological group following the method of Krull, who noticed that the Galois groups of A with respect to all finite abelian extension fields of K form a complete system of neighborhoods for the unit element of G. This topological group is compact and therefore the results of Pontrjagin concerning abelian compact groups can be applied to it. (2) The fundamental group I consisting of the so-called "idèles" of K. The introduction of the idèles enables the author to dispose entirely of the congruence groups [see also C. Chevalley, J. Math. Pures Appl. (9) **15**, 359–371 (1936)]. An idèle is a sequence of non-zero $\mathfrak{p}$-adic numbers (where $\mathfrak{p}$ runs through all different finite and in-

finite prime ideals of K taken in some fixed order), of which only a finite number are not $\mathfrak{p}$-adic units. The product of two idèles is obtained by multiplying the coordinates. If α is any non-zero number of K, the sequence all of whose elements are α is an idèle since there is only a finite number of $\mathfrak{p}$'s for which α is not a $\mathfrak{p}$-adic unit. These special idèles form a subgroup P of all idèles, the principal idèles. It is isomorphic with the multiplicative group of all non-zero numbers of K. For the idèles a norm is defined which satisfies the usual axioms and which agrees with the definition of norm for numbers. Furthermore, a topology is introduced in I which, though weaker than the $\mathfrak{p}$-adic topology, is more suitable for the present purposes. The character group of I is of great importance. In particular, a character of I which is identically equal to 1 on P is called a differential in analogy to the differentials in the case of algebraic function-fields [see A. Weil, J. Reine Angew. Math. **179**, 129–133 (1938)]. The main result is the proof of the existence of an isomorphism between the character group of G and the group of differentials of K. This isomorphism can be constructed explicitly. Since, according to Pontrjagin, the character group of the compact group G determines the group G itself, the group G therefore also is determined by K and can indeed be constructed explicitly. This last fact can be considered as the content of Artin's general law of reciprocity. *O. Todd-Taussky* (London).

Hecke, E. Die Klassenzahl imaginär-quadratischer Körper in der Theorie der elliptischen Modulfunktionen. Monatsh. Math. Phys. **48**, 75–83 (1939). [MF 627]

This article begins with a recapitulation of the different known relations between imaginary quadratic fields and modular functions ("complex multiplication," "binary theta-series") and then proceeds to add a new occurrence of the class-number of imaginary quadratic fields in the theory of modular functions. Throughout this paper the author makes use of notations and results of his previous publications on modular forms and Euler products [cf., in particular, Math. Ann. **114**, 1–28, 316–351 (1937)].

Let q be an odd prime number not less than 5 and $f^{(\nu)}(\tau)$ ($\nu=1, 2, \cdots, \kappa$) the system of linearly independent cusp-forms (that is, modular forms vanishing in the parabolic points of the fundamental region) of dimension $-k$ belonging to the group $\Gamma_0(q)$ with $c\equiv 0 \pmod q$ in

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(on p. 77 this invariance with respect to $\Gamma_0(q)$ is expressed in terms of the quoted papers as follows: the forms $f^{(\nu)}(\tau)$ are of "stufe q" (that is, belong to the principal congruence group $\overline{\Gamma}(q)$), of "divisor q," "normalized" and of "character $\epsilon(n)=1$"). The linear set generated by the $f^{(\nu)}(\tau)$ goes over into itself under the linear operators T_n and $T_m{}^q$ defined in the second paper quoted above. The automorphisms induced by the $T_m{}^q$ lead to a representation by matrices $\lambda(m)$ of degree κ. The $\lambda(m)$ are connected with the functions $f^{(\nu)}(\tau)$ by the matric equation

$$\sum_{\nu=1}^{\kappa} f^{(\nu)}(\tau)B^{(\nu)}=\sum_{m=1}^{\infty} \lambda(m)e^{2\pi i m\tau}$$

with certain constant matrices $B^{(\nu)}$. The law of composition of the $\lambda(m)$ finds its expression in the formula

$$\sum_{m=1}^{\infty} \lambda(m)m^{-s}=(1-\lambda(q)q^{-s})^{-1}\prod_{p\neq q}(1-\lambda(p)p^{-s}+p^{1-2s})^{-1}.$$

Now the problem of the present paper is the determination of the characteristic roots of $\lambda(q)$ and of its trace $S(\lambda(q))$. For $k=2$ the result is as follows: the characteristic roots are only ± 1, and

$$S(\lambda(q))=\tfrac{1}{2}\delta_q\cdot h(4q)-1,$$

$h(4q)$ being the class-number of primitive positive binary quadratic forms of determinant $-4q$, and $\delta_q=2$ or $4/3$ for $q\equiv 7$ or $3 \pmod 8$, respectively, and $\delta_q=1$ otherwise.

The method consists in singling out those forms which belong also to the enlarged group $\Gamma^*(q)=\Gamma_0(q)+\Gamma_0(q)H$, where H is the substitution $\tau\to -1/q\tau$. It is shown that the number of linear independent ones of these forms is equal to the number of the negative characteristic roots of $\lambda(q)$, a result which is derived for $k<12$ and $k=14$ since for these dimensions $-k$ there exist no cusp-forms that belong to the full modular group. For $k=2$ the number κ is equal to the genus $p_0(q)$ of $\Gamma_0(q)$, and the number of linear independent forms belonging to $\Gamma^*(q)$ is similarly equal to the genus $p^*(q)$ of $\Gamma^*(q)$. Now Fricke has proved that

$$p^*(q)=\tfrac{1}{2}p_0(q)+\tfrac{1}{2}-\tfrac{1}{4}\delta_q\cdot h(4q),$$

and this yields the above-mentioned result. For $k=4$, 6, 8, 10, 14 the enumerations can be done by means of the Riemann-Roch theorem, and the result is

$$S(\lambda(q))=(-q)^{k/2-1}\cdot h\cdot \rho_q,$$

with h denoting the class-number of the quadratic field $K(\sqrt{-q})$ and $\rho_q=1$, 2, 1/2 if the quadratic residue symbol $(-q/2)$ is 1, -1, 0, respectively. Analogous results for the other values of the dimension $-k$ are indicated and the connection with a theorem of Petersson concerning the simultaneous transformations of the $\lambda(m)$ into diagonal form is mentioned. *H. Rademacher* (Philadelphia, Pa.).

Humbert, Pierre. Sur les nombres de classes de certains corps quadratiques. Comment. Math. Helv. **12**, 233–245 (1940). [MF 1572]

The object of this paper is to show how one can construct imaginary quadratic fields having a class of ideals of given order g; the class number h of such a field will then be divisible by g. In particular it is shown that for a given g there exists an infinitude of such fields. To do this the author proves the following lemma: If P, a are relatively prime, the field $R((a^2-4P^g)^{\frac{1}{2}})$ contains an ideal whose gth power and no lower is a principal ideal, provided the discriminant of the field is not greater than $-4P^{g/q}$ (q is the smallest prime factor of g). The proof consists in setting up such an ideal by entirely elementary methods. The lemma is then used to construct infinitely many such fields. This construction is especially simple if g is an odd prime; for example, if $g=3$ we can use $a=2$, $P=$any number of the form $24k+3$ such that $P-1$ is square-free (for example, $P=3$ gives us $R(\sqrt{-26})$ for which $h=6$). For real quadratic fields a theorem similar to the lemma just quoted can be proved, but it seems to be impossible to prove the existence of the desired fields in this manner. As a by-product the author obtains the fact that $4P^4+1$ is not a prime for $P>1$, but he does not notice that this is trivial since $4P^4+1=(2P^2+2P+1)(2P^2-2P+1)$.
H. W. Brinkmann (Swarthmore, Pa.).

Hasse, H. Produktformeln für verallgemeinerte Gausssche Summen und ihre Anwendung auf die Klassenzahlformel für reelle quadratische Zahlkörper. Math. Z. **46**, 303–314 (1940). [MF 2400]

In spite of the great elegance of the explicit formula for the class number h of a quadratic field, the actual compu-

tation of h on the basis of this formula meets with considerable difficulties in the case of a real quadratic field. The author shows that, if the discriminant is a positive prime p, the expressions appearing in the formula can be obtained by the use of a product formula for Gaussian sums. He derives this product formula in a far more general form than needed for this particular purpose. Let G be the multiplicative group of a Galois field with $q=p^s$ elements. To every character ψ of G there corresponds a Gaussian sum $\tau(\psi)=\sum_x\psi(x)e(x)$, where x ranges over the elements of G, and where $e(x)=e^{(2\pi i/p)\operatorname{tr}(x)}$, $\operatorname{tr}(x)$ denoting the trace of x. Let n be the order of ψ and let m be a divisor of $(q-1)/n$. A character ψ_m of the subgroup G^m of G can be defined by the formula $\psi_m(x^m)=\psi(x)$, x in G; we set $\psi_m(y)=0$ if y is not an mth power in G. The generalized Gaussian sum corresponding to G and ψ_m is defined as the expression $\tau(\psi_m)=\sum_x\psi_m(x)e(x)$ and, for any a in G, we set $\tau_a(\psi_m)=\sum_x\psi_m(x)e(ax)$. Let k be a divisor of m, $m=kl$; let $\mathfrak{W}: w_1, w_2, \cdots, w_k$ be a system of representatives for the residue classes of $G \bmod G^k$, $w_1=1$, and let $\mathfrak{R}$ be a system of representatives for the residue classes of G mod G^m. If $\mathfrak{r}_i$ denotes a row consisting of the l elements of $\mathfrak{R}$ which are congruent to w_i (mod G^m), then we have formulae $w_i\mathfrak{r}_1=\mathfrak{r}_iW_i$, where W_i is a monomial matrix with coefficients in G^m; the product of the l non-vanishing coefficients of W_i will be denoted by $|W_i|$. The products studied in the paper are $\Theta_{\mathfrak{r}_i}(\psi_m)=\Pi\tau_{r_i}(\psi_m)$, where r_i ranges over the elements of $\mathfrak{r}_i$. The main result is given by the formula

$$\Theta_{\mathfrak{r}_i}(\psi_m)=\frac{1}{k}\sum_{\chi^k=\psi}P_i(\chi,\psi_m)\tau(\chi).$$

Here the sum is extended over the k characters of G whose kth power is equal to ψ. The coefficient $P_i(\chi,\psi_m)$ is defined by

$$P_i(\chi,\psi_m)=\sum_{j=1}^{k}\chi\left(\frac{w_j}{w_i}\right)\psi_m\left(\frac{|W_i|}{|W_j|}\right)\sum_{\mathfrak{r}_jX=1}\psi_m(|X|),$$

where X ranges over all columns consisting of l elements of G^m which satisfy the condition $\mathfrak{r}_iX=1$; the product of the l elements of X is denoted by $|X|$. For the application to class numbers of quadratic fields mentioned above, only the case $n=2$, $q=p$, $m=(p-1)/2$, $k=2$ is needed. The author surmises that the general formula has a corresponding significance for the class numbers of real, absolutely abelian fields. The special case $k=1$ of the formula can be used for the determination of the sign of the quadratic Gaussian sum. The author believes that his formula might also be applied in order to find the still unknown position of the higher Gaussian sums in the complex plane.

R. Brauer (Toronto, Ont.).

Vinogradow, I. M. A general property of prime numbers distribution. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 365–372 (1940). (Russian. English summary) [MF 2797]

Let $0<\alpha<1$, $N\geqq 2$. An estimate is obtained for the sum $\sum e^{2\pi i k p^\alpha}$, where p ranges over the primes not greater than N, and $k\ll N^{\alpha/3}$ is a positive constant. Let $0<\sigma<1$, $b>0$, $\epsilon>0$. It is proved that the number T of primes not greater than N such that $0\leqq bp^\alpha-[bp^\alpha]<\sigma$ is given by the formula $T=\sigma\pi(N)+O(N^{1+\epsilon}\Delta)$, $\Delta=(bN^{\alpha-1}+b^{-1}N^{-\alpha}+N^{-2\alpha/5})^{1/5}$. Otherwise stated, set $\beta=b^{-m}$, $m=1/\alpha$; T is the number of primes not greater than N in the intervals $\beta c^m\leqq p<\beta(c+\sigma)^m$, for integers c.

G. Pall (Princeton, N. J.).

Vinogradow, I. On the estimations of some simplest trigonometrical sums involving prime numbers. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 371–398 (1939). (Russian. English summary) [MF 2085]

Let $e(x)=e^x$. Let p range over the primes in the interval $N-A<p\leqq N$; n and k denote fixed positive integers; ϵ, η, h given positive numbers; a/q a rational fraction in lowest terms, $q>0$; a real λ written $=a/q+\theta/q^2$, $|\theta|\leqq 1$; $N>2$, $r=\log N$; $1\leqq A\leqq N$. Five theorems are proved, Theorem 1 and a modified Theorem 2 having been previously announced [cf. these Rev. 1, 293]. Theorem 1: let $\eta<\epsilon\leqq\frac{1}{4}$; $A\gg Ne(-r^{1-2\epsilon})$, $q\leqq e(r^\epsilon)$; then $\sum_pe(2\pi ip^na/q)\ll A/(r^{1-3\epsilon}q^{\frac{1}{2}-\eta})$. Theorem 2: let ϵ, η, $h<1$; $q>e(r^\epsilon)$, $A\geqq N^{h+2/3}q^{3/2}$; then $\sum_pe(2\pi ip^na/q)\ll Aq^{\eta-\frac{1}{2}}$. Theorem 3: let $h\leqq 1/6$, $q\leqq N$; set $\tau=6+(\log r)/(\log \overline{1+h})$, $L=kN^{(2+h)/3}A^{-1}+NqA^{-2}+kq^{-1}+k^4q^{-2}$; then $\sum_pe(2\pi i\lambda kp)\ll Ar^\tau L^{1/2}$. Write $\{x\}=x-[x]$. Theorem 4: let ϵ, $h\leqq 1/6$, $0<\delta\leqq 1$, $N\geqq N_0$, N_0 sufficiently large; $q\leqq N$; let T denote the total number of values of p, and T_1 the number with $0\leqq\{\lambda p\}\leqq\delta$; then $T_1=\delta T+O(A\Delta)$, where $\Delta=e(r^\epsilon)K^{1/2}$, $K=N^{h+2/3}A^{-1}+N^{1+h}q^{1-h}A^{-2}+q^{h-1}$. Theorem 5: let $\eta<1$, $1\leqq Q\leqq N$, $0\leqq\delta\leqq 1$; let T_2 denote the number of values p for which the residue P of p (mod Q) lies in $0\leqq P\leqq\delta Q$; then $T_2=\delta T+O(A\Delta)$, where $\Delta=N^\eta H^{1/2}$, $H=N^{2/3}A^{-1}+NQA^{-2}+Q^{-1}$. Corrections are given on p. 396 to the author's paper on estimation of trigonometric sums [same journal **1938**, no. 5–6]: the limits for k should be $k<1+R/(-\log \overline{1-\nu})$, with $R=\log U$ in Lemma 5, $R=\log 2Z$ in Lemma 6; the limit to K in Lemma 6 should be $2^{bn/6}p_1^{bk}(p_1\cdots p_k)^{-b}$; $1/\rho$ in Theorem 2 should be $(n+1)^3\log 2(n+1)$.

G. Pall (Princeton, N. J.).

Segal, B. On integers of standard form of a definite type. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 519–538 (1939). (Russian. English summary) [MF 2094]

Let $p_1, \cdots, p_l$ be distinct given primes, $l\geqq 2$, $n_1, n_2, \cdots$ the sequence of integers of the form $p_1^{\alpha_1}\cdots p_l^{\alpha_l}$, in order of magnitude. Then $n_{j+1}/n_j\to 1$. This remains true if $l\geqq 3$ and the α_i are restricted to primes. For $l=2$ Segal had proved [C. R. (Doklady) Acad. Sci. URSS (A) **1933**, 39–44] Theorem 1: let I_N be the number of solutions of $N=dp_1^{\alpha_1}p_2^{\alpha_2}$ with positive integers α_i, and $1\leqq d\leqq e^{\Delta_1}$, $\Delta_1\log\log N=\log\log\log N$; then

$$I_N=\Delta_1(\log N)/\beta_2+O(\log N/\log\log N).$$

Here $\beta_l=\log p_1\cdots\log p_l$. Using a result of A. Gelfond [same vol., 509–518; cf. these Rev. 1, 295], Δ_1 can be replaced by $\Delta=e^{-\rho}$, $\rho=(\log\log N)^{\frac{1}{2}-\epsilon}$, $\epsilon>0$; and Segal proves Theorem 2: Let $l\geqq 3$, $I_{N,l}$ the number of solutions of $N=dp_1^{\alpha_1}\cdots p_l^{\alpha_l}$ with positive integral α_i and $e^{-\Delta}\leqq d\leqq e^\Delta$; then

$$I_{N,l}=E+O((\log N)^{l-1}e^{-\rho_0}),$$

where $E=(2/(l-1)!)(\log N)^{l-1}\Delta/\beta_l$, $\rho_0=(\log\log N)^{\frac{1}{2}-\epsilon_0}$, $0<\epsilon_0<\epsilon$. Using also results by Vinogradow [see the preceding review], Segal restricts the α_i to be primes, when the asymptotic formula becomes $E(\log\log N)^{-l}(1+O(\Delta_1))$. A similar result is given for $N=dp_1^\pi p_2^\alpha$, where π runs over primes, α over positive integers.

G. Pall.

Hua, Loo-Keng. Sur une somme exponentielle. C. R. Acad. Sci. Paris **210**, 520–523 (1940). [MF 2259]

Let $f(x)=a_kx^k+\cdots+a_1x+a_0$, $(q, a_1, \cdots, a_k)=1$,

$$S_q=\sum_{x=1}^{q}\exp(2\pi if(x)/q).$$

A neat proof is sketched that $S_q < Cq^{1+\epsilon-1/k}$, where $C=C(k, \epsilon)$. Mordell [Quart. J. Math. 3, 161–167 (1932)] proved $S_q = O(q^{1-1/k})$ when q is a prime. Hua treated this problem earlier [J. London Math. Soc. 13, 54–61 (1938)]. This result fills a lacuna in an article by Vinogradow [Rec. Math. [Mat. Sbornik] N.S. 3 (45), 453 (1938)] and has applications to the theory of uniform distributions of the fractional part of $\alpha f(x)$, and to the Waring problem for polynomials [cf. the author's paper; these Rev. 2, 35]. *G. Pall.*

van Veen, S. C. On the sums $\sum_{p\leq x} \log p/p$ and $\sum_{p\leq x} 1/p$. Mathematica, Zutphen. B. 8, 135–145 (1940). (Dutch) [MF 1202]

Elementary proofs (not containing anything from the theory of functions of a complex variable) of the following (known) formulae ($x \geqq 2$):

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + r(x), \quad \sum_{p\leq x} \frac{1}{p} = \log\log x + B + R(x),$$

where p runs through all primes not greater than x and

$$|r(x)| < 3, \quad |R(x)| < \frac{6}{\log x}, \quad B = C + \sum_{n=1}^{\infty}\left\{\log\left(1-\frac{1}{p_n}\right)+\frac{1}{p_n}\right\}$$

(p_n is the nth prime and C is Euler's constant).

H. D. Kloosterman (Leiden).

Levinson, Norman. On Hardy's theorem on the zeros of the zeta function. J. Math. Phys. Mass. Inst. Tech. 19, 159–160 (1940). [MF 2639]

The author deduces from Riemann's formula

$$z^{\frac{1}{4}}\frac{d}{dz}\{z^{\frac{3}{4}}\theta'(z)\} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\Xi(2t)z^{-it}dt, \qquad \Re z > 0,$$

$$\theta(z) = \sum_{-\infty}^{\infty} e^{-n^2\pi z}, \quad \Xi(t) = -\tfrac{1}{2}(\tfrac{1}{4}+t^2)\pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\zeta(\tfrac{1}{2}+it),$$

and the known behavior of $\theta(z)$ near $z=i$ and $z=2i$ that

$$\int_{-\infty}^{\infty}\Xi(2t)e^{\frac{1}{2}(\pi-\epsilon)t}dt \to 0, \quad \left|\int_{-\infty}^{\infty}\Xi(2t)2^{-it}e^{\frac{1}{2}(\pi-\epsilon)t}dt\right| \to \infty,$$

when $\epsilon \to +0$. But this is obviously impossible if $\Xi(2t)$ (which is even, continuous and real for real t) has only a finite number of changes of sign in $(-\infty, \infty)$.

A. E. Ingham (Cambridge, England).

Avakumović, Vojislav G. Neuer Beweis eines Satzes von G. H. Hardy und S. Ramanujan über das asymptotische Verhalten der Zerfällungskoeffizienten. Amer. J. Math. 62, 877–880 (1940). [MF 2887]

Let $p(n)$ denote the number of unrestricted partitions of a positive integer n. The asymptotic formula of Hardy and Ramanujan to which reference is made in the title of the paper is

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}, \qquad n\to\infty.$$

[In the paper $4n\sqrt{3}$ is misprinted as $4\sqrt{3n}$.] Hardy and Ramanujan [Proc. London Math. Soc. (2) 17, 75–115 (1918)] state that they were unable at that time to obtain, by any method which did not depend on Cauchy's theorem, a result as precise as this. The new short proof by Avakumović does not use Cauchy's theorem, but requires some knowledge of the convergence of infinite integrals and the inversion of repeated infinite integrals. *R. D. James.*

van Kampen, E. R. and Wintner, Aurel. On the almost periodic behavior of multiplicative number-theoretical functions. Amer. J. Math. 62, 613–626 (1940). [MF 2458]

A function $f(n)$ is multiplicative if $f(n_1n_2)=f(n_1)f(n_2)$ whenever $(n_1, n_2)=1$ and $f(1)=1$. If p is a fixed prime number, a function $g(n)=g_p(n)$ is "prime" if $g(np^k)=g(p^k)$ whenever $(n, p)=1$. Any $f(n)$ is a product $\Pi_p g_p(n)$. The authors prove theorems of the following type. A prime function $g(n)$ is almost periodic (B) if and only if $\sum p^{-k}|g(p^k)|<\infty$. Also, the product of a finite number of such functions $g(n)$ which belong to distinct prime numbers, and for which $g(1)=1$, is almost periodic (B) if each factor is. A multiplicative function $f(n)\geqq 0$ is almost periodic (B) if $\sum_{n=1}^{\infty}|h(n)|<\infty$, where $h(n)$ is the multiplicative function for which $h(p^k)=p^{-k}\{f(p^k)-f(p^{k-1})\}$. Also the Fourier series of $f(n)$ is $\sum_{m=1}^{\infty} a_m c_m(n)$, where $a_m=\sum_{l=1}^{\infty} h(ml)$ and

$$c_m(n) = \sum_s \exp\left(2\pi i\frac{s}{m}n\right), \quad 1\leqq s\leqq m;\ (s, m)=1.$$

For instance,

$$\frac{\sigma_\alpha(n)}{n^\alpha} \sim J(\alpha+1)\sum_{m=1}^{\infty}\frac{c_m(n)}{m^{\alpha+1}}, \qquad \alpha>0.$$

S. Bochner (Princeton, N. J.).

van Kampen, E. R. On uniformly almost periodic multiplicative and additive functions. Amer. J. Math. 62, 627–634 (1940). [MF 2459]

The author discusses uniform almost periodicity (Bohr) of a function $f(n)$ which is multiplicative [see the preceding review] or of a function $g(n)$ which is additive, that is, $g(n_1n_2)=g(n_1)+g(n_2)$ if $(n_1, n_2)=1$ and $g(1)=0$. Purely formally, just as $f(n)$ is an infinite product of factors belonging to different primes, so $g(n)$ is an infinite sum of such terms. The author's main result is that $f(n)$ is u.a.p. if and only if the infinite product is uniformly convergent and that $g(n)$ is u.a.p. if and only if the infinite sum is uniformly convergent. Furthermore, in the case of additive functions the result also holds for functions which are complex-valued.

S. Bochner (Princeton, N. J.).

Erdös, Paul and Wintner, Aurel. Additive functions and almost periodicity (B^2). Amer. J. Math. 62, 635–645 (1940). [MF 2460]

For additive functions [see the preceding review] the authors prove a clear-cut criterion of almost periodicity (valid also for complex-valued functions) which apparently has no analogue for the more difficult type of multiplicative functions [see the second review above]. The theorem states that an additive function $f(n)$ is almost periodic (B^2) if and only if both series

$$\sum_p \frac{f(p)}{p}, \quad \sum_{p,l}\frac{|f(p^l)|^2}{p^l}$$

are convergent (the letter p to run over all primes, and the exponent l over all positive integers). The proof includes interesting points. *S. Bochner* (Princeton, N. J.).

Hartman, Philip and Wintner, Aurel. On the almost periodicity of additive number-theoretical functions. Amer. J. Math. 62, 753–758 (1940). [MF 2876]

It was known [cf. the preceding review] that an additive function $f(n)$, $n=1, 2, \cdots$, is almost periodic (B) if and only if the two series

$$\sum_p p^{-1}f(p), \quad \sum_{l=1}^{\infty}\sum_p p^{-l}|f(p^l)|^2$$

are convergent, p=prime number. The authors obtain the same conclusion replacing those two series by the four series

$$\sum_p p^{-1}f(p), \quad \sum_p p^{-1}|f^+(p)|^2,$$

$$\sum_{l=2}^{\infty}\sum_p p^{-l}|f(p^l)|, \quad \sum_{|f(p)|\geqq 1} p^{-1}|f(p)|.$$

In this connection, $f^+(p)=f(p)$ if $|f(p)|<1$ and $=1$ if $|f(p)|\geqq 1$. *S. Bochner* (Princeton, N. J.).

Hartman, Philip and Kershner, Richard. On upper limit relations for number theoretical functions. Amer. J. Math. 62, 780–786 (1940). [MF 2878]

The proof of a result of the form

$$\limsup_{x\to\infty} f(x)g(x)=1,$$

where $f(x)$ is a number theoretical function and $g(x)$ is elementary, is often made by first proving $\lim_{n\to\infty} f(r_n)g(r_n)=1$, where r_n is the product of the first n primes. It is then necessary to show that the second result implies the first. The aim of this paper is to find general conditions under which this implication is valid. The main result is a theorem which gives conditions that are sufficient to imply

$$\limsup_{x\to\infty} f(x)g(x)=1,$$

where $f(x)$ is additive and $g(x)=\log\log x/\log x$. Similar results are given for the case where $f(x)$ is strongly additive or strongly multiplicative and $g(x)$ is a non-increasing function. These theorems are based on a lemma which gives conditions sufficient to imply

$$\limsup_{x\to\infty} f(x)g(x)=1$$

from $\lim_{n\to\infty} f(r_n)g(r_n)=1$, where the r_n form an increasing sequence of integers. *H. S. Zuckerman* (Seattle, Wash.).

Hartman, Philip and Wintner, Aurel. On the standard deviations of additive arithmetical functions. Amer. J. Math. 62, 743–752 (1940). [MF 2875]

The existence of an asymptotic distribution function is in general not sufficient to insure almost periodicity. However, in the case of arithmetical functions this is enough. The authors prove that an additive function $f(n)$ is almost periodic (B^2) if and only if its asymptotic distribution function possesses a second moment

$$\int_{-\infty}^{\infty} x^2 d\sigma(x)<\infty.$$

S. Bochner (Princeton, N. J.).

Erdös, P. and Kac, M. The Gaussian law of errors in the theory of additive number theoretic functions. Amer. J. Math. 62, 738–742 (1940). [MF 2874]

If an arithmetical function satisfies $f(m)+f(n)=f(mn)$ for $(m,n)=1$ and $f(p^\alpha)=f(p)$, $|f(p)|\leqq 1$ for all prime numbers p, put

$$A_n=\sum_{p<n} p^{-1}f(p), \quad B_n^2=\sum_{p<n} p^{-1}f^2(p).$$

Suppose that $B_n\to\infty$ as $n\to\infty$. If K_n denotes, for a real ω, the number of integers m between l and n for which $f(m)<A_n+\omega\sqrt{2}B_n$, then

$$\lim_{n\to\infty}\frac{K_n}{n}=\pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\exp(-u^2)du.$$

The analogy of this theorem to a classical statistical limit theorem is obvious. The proof consists in the justification of the interchange of two limiting processes. The one is $n\to\infty$ of the theorem; the other consists in letting $l\to\infty$ in the case when only the first l primes are considered. The interchange is based on some of the deeper "elementary" arithmetical methods, in particular on a method of Vigo Brun's classical memoir concerning the sieve of Eratosthenes.

E. R. van Kampen (Baltimore, Md.).

Sambasiva Rao, K. On the representation of a number as the sum of the kth power of a prime and an lth power-free integer. Proc. Indian Acad. Sci., Sect. A. 11, 429–436 (1940). [MF 2604]

The author considers the representation of an integer n in the respective forms $n=p^k+g_t$, $n=\psi(p)+g_t$, where p is a prime, g_t is an integer not divisible by the tth power of any integer greater than 1 and $\psi(p)$ is an integral valued polynomial in p of degree k with leading coefficient positive. Asymptotic formulas for the numbers of solutions are obtained for $t\geqq k$ in the first case and $t>k$ in the second. The cases $k=1$, $t=2$ and $k=t$ had been considered by previous writers using similar methods. Their results were less precise because the Page-Walfisz-Siegel theorem on primes in an arithmetical progression was not then known. The present paper makes full use of this. *R. D. James.*

Sambasiva Rao, K. Generalisation of a theorem of Pillai-Selberg. Proc. Indian Acad. Sci., Sect. A. 11, 502–504 (1940). [MF 2689]

The following theorem is established: Let $N_{r,t}^k(x)$ denote the number of kth power-free integers not exceeding x, the number of whose prime factors (taking into consideration the degree of multiplicity) is congruent to t modulo r. Then $N_{r,t}^k(x)\sim x/\zeta(k)$, where $\zeta(k)=\sum_{n=1}^{\infty}(1/n^k)$. Special cases of this result are due to von Mangoldt, Pillai [Proc. Indian Acad. Sci., Sect. A. 11, 13–20 (1940); these Rev. 1, 293] and Selberg [Math. Z. 44, 306–318 (1939)]. The main point of the proof is to show that $L_r^{(k)}(x)=O(x)$, where $L_r^{(k)}(x)=\sum\lambda_r(n)$, summed over all kth power-free integers not exceeding x; $\lambda_r(n)=\exp(2\pi it(n)s/r)$, $s=1,\cdots,r-1$; and $t(n)$ is the number of prime factors of n, each prime factor being counted according to its degree of multiplicity. This follows at once from the corresponding result in the paper of Pillai referred to above. *R. D. James.*

Ostmann, Hans-Heinrich. Über die Dichte der Summe zweier Zahlenmengen. Deutsche Math. 5, 177–212 (1940). [MF 3116]

Let A and B be two sets of non-negative integers such that $0\epsilon A$ and $0\epsilon B$. Let α and β denote respectively the densities of A and B. (The densities are meant in the sense of Schnirelman.) By $A(n)$ and $B(n)$ one understands the number of elements of A or B which are not greater than n. The arithmetic sum $A+B$ is the set of integers of the form $a+b$, where $a\epsilon A$ and $b\epsilon B$. Let furthermore $J(n)=A(n)+B(n)-n$. Two cases are possible: (1) $J(n)\geqq 0$ for $n=1, 2, \cdots$. (2) There exists a least integer $j(\geqq 3)$ such that $J(j)=-1$. In the first case the density γ of $A+B$ is 1. In the second case the author obtains the estimate

$$\gamma\geqq\alpha+\beta-\frac{1}{j+2}.$$

More general theorems are proved but their statements are too complicated to be reproduced here. *M. Kac.*

ANALYSIS

Special Functions

Lehmer, D. H. On the maxima and minima of Bernoulli polynomials. Amer. Math. Monthly **47**, 533–538 (1940). [MF 3078]

Let M_n and m_n denote the maximum and minimum of the Bernoulli polynomial $B_n(x)$ for $0 \leqq x \leqq 1$. Then for n even

$$M_{4h} = B_{4h}(\tfrac{1}{2}) = (1-2^{1-4h})|B_{4h}|, \qquad h>0,$$
$$M_{4h+2} = B_{4h+2}(0) = B_{4h+2},$$
$$m_{4h} = B_{4h}(0) = -|B_{4h}|, \qquad h>0,$$
$$m_{4h+2} = B_{4h+2}(\tfrac{1}{2}) = -(1-2^{-1-4h})B_{4h+2}.$$

When n is odd, an exact formula is not available because no exact formula exists for the roots r_{2k} of $B_{2k}(x)=0$. In this case an asymptotic formula is obtained for the root, by means of which it is shown that

$$-m_{2k+1} = M_{2k+1} < \frac{2k+1}{2^{4k-2}}\left\{|E_{2k}| + \frac{2}{\pi}(1-2^{-2k+1})|B_{2k}|\right\},$$

where E_n is the nth Eulerian number. The inequality $M_n < 2n!/(2\pi)^n$ holds for all n not of the form $4h+2$, while the inequality $m_n > -2n!/(2\pi)^n$ holds for all n not of the form $4h$. In these exceptional cases one must supply the factor $\zeta(n) = \sum_{v=1}^{\infty} v^{-n}$, and replace the inequality sign by equality. The paper closes with a ten place table of roots r_{2k} and corresponding M_{2k+1}. *W. E. Milne.*

Shastri, N. A. On Angelescu's polynomial $\pi_n(x)$. Proc. Indian Acad. Sci., Sect. A. **11**, 312–317 (1940). [MF 2746]

Shastri, N. A. Some results involving Angelescu's polynomial $\pi_n(x)$. Proc. Indian Acad. Sci., Sect. A. **12**, 73–82 (1940). [MF 2754]

The generating function

$$\phi\left(\frac{t}{1-t}\right)\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} \pi_n(x)t^n$$

defines certain polynomials $\{\pi_n(x)\}$ which are generalizations of the Laguerre polynomials. Various formal properties of the polynomials $\pi_n(x)$ are investigated, especially their relation to Laguerre, Hermite and other polynomials as well as to Whittaker's and Bessel's functions. *G. Szegö* (Stanford University, Calif.).

Bateman, H. The polynomial of Mittag-Leffler. Proc. Nat. Acad. Sci. U. S. A. **26**, 491–496 (1940). [MF 2635]

A discussion of the polynomial

$$g_n(z) = 2zF(1-n, 1-z; 2; 2)$$

which occurs in the expansion

$$(1+t)^z(1-t)^{-z} = 1 + \sum g_n(z)t^n, \qquad |t|<1,$$

which played a role in Mittag-Leffler's investigations on the analytic representation of the integrals and invariants of a linear homogeneous differential equation as well as in his later researches on the analytical continuation of a power series in the interior of its principal star. The author derives recurrence relations for these polynomials and some generalizations, expressions in terms of hypergeometric series and definite integrals, estimates for large values of n, relations to Laguerre polynomials, numerical values of $g_n(m)$, etc. *E. Hille* (New Haven, Conn.).

Kober, H. On some generalisations of Laguerre polynomials. Proc. Edinburgh Math. Soc. (2) **6**, 135–146 (1940). [MF 2844]

The author considers certain modifications of the Laguerre polynomials with a view of obtaining complete systems in $L_q(0, \infty)$ when $\Re(\alpha)$ is negative. Let

$$E_m(x) = e^{-x} - \sum_{h=0}^{m-1} (-x)^h/h!,$$

$$L_{n,m}^{(\alpha)}(x) = e^{\frac{1}{2}x} \sum_{h=0}^{n} \binom{n+\alpha}{n-h} \frac{(-x)^h}{h!} E_{m-h}(\tfrac{1}{2}x),$$

$$\psi_{n,m}^{(\alpha)}(x) = e^{-\frac{1}{2}x}x^{\frac{1}{2}\alpha}L_{n,m}^{(\alpha)}(x).$$

Let $1 \leqq p \leqq \infty$, $1/p+1/q=1$. Let $\Re(\alpha) \neq 2/p$ be neither a negative even integer nor zero, and let $j=j(\alpha, p)$ be an integer defined by $-2j-2/p < \Re(\alpha) < -2j-2/p+2$, when $\Re(\alpha) < -2/p$, otherwise 0. When $j>0$ the system $\{\psi_{n,j}^{(\alpha)}(x)\}$ $(n=0, 1, 2, \cdots)$ belongs to $L_p(0, \infty)$ and is complete with respect to $L_p(0, \infty)$. If $p<\infty$, it is also closed in $L_p(0, \infty)$. If $j>0$ and $r+s$ is odd, $\psi_{r,j}^{(\alpha)}(x)$ and $\psi_{s,j}^{(\alpha)}(x)$ are orthogonal on $(0, \infty)$. Generating functions and numerous recurrence relations are derived as well as relations to Hankel and other transforms. *E. Hille* (New Haven, Conn.).

Erdélyi, Artur. Einige nach Produkten von Laguerre'schen Polynomen fortschreitende Reihen. Akad. Wiss. Wien, S.-B. IIa **148**, 33–39 (1939). [MF 2537]

Sansone, G. I polinomi di Hermite e di Laguerre come autosoluzioni. Boll. Un. Mat. Ital. (2) **2**, 193–200 (1940). [MF 2972]

The author gives short, elementary and elegant proofs of the following theorems. I. Let $v(x) \not\equiv 0$ be a solution of $v''-2xv'+2\lambda v=0$, λ arbitrary complex number, such that $|v(x)| \leqq M \exp[kx^2]$ for some k, $0 \leqq k < 1$, and $|x| \geqq x_0$. Then λ must be a non-negative integer n and $v(x)$ is a constant multiple of the nth polynomial of Hermite. II. Let $v(x) \not\equiv 0$ be a solution of $xv''+(\alpha-x+1)v'+\lambda v=0$, $\alpha>-1$, λ arbitrary complex number, such that $|v(x)| \leqq M \exp(kx)$, $0 \leqq k < 1$, and $\lim_{x\to+0} x^{\alpha+1}v'(x)=0$, then λ must be a non-negative integer n and $v(x)$ is a constant multiple of the nth polynomial of Laguerre $L_n^{(\alpha)}(x)$. *E. Hille.*

Feldheim, E. Expansions and integral-transforms for products of Laguerre and Hermite polynomials. Quart. J. Math., Oxford Ser. **11**, 18–29 (1940). [MF 1863]

Author proves that the coefficients in the two expansions

$$L_m^{(\alpha)}(x)L_n^{(\beta)}(x) = \sum_{s=0}^{m+n} b_s \frac{x^s}{s!}$$

and

$$L_m^{(\beta-m+n)}(x)L_n^{(\alpha+m-n)}(x) = (-)^{m+n}\sum_{s=0}^{m+n} b_s L_s^{(\alpha+\beta)}(x)$$

are the same. Further, he gives an expansion of a product of three Hermite polynomials in a series of Hermite polynomials. In the second part of the paper integral representations and integral formulae for products of Hermite polynomials are obtained. Among others there is the formula

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x+y)H_n(x-z)dx = \pi^{\frac{1}{2}}n!2^m y^{m-n}L_n^{(m-n)}(2yz), \qquad m \geqq n.$$

Many of the results of the second part are particular cases

of the relation (given in a postscript without proof)

$$\int_{-\infty}^{+\infty} e^{-x^2/a} H_m\left(\frac{x+y}{\lambda}\right) H_n\left(\frac{x+z}{\mu}\right) dx$$
$$= (a\pi)^{\frac{1}{2}}\left(1-\frac{a}{\lambda^2}\right)^{\frac{1}{2}m}\left(1-\frac{a}{\mu^2}\right)^{\frac{1}{2}n} \sum_{k=0}^{\min(m,n)} \left\{\frac{4a^2}{(\lambda^2-a)(\mu^2-a)}\right\}^{\frac{1}{2}k}$$
$$\times k!\binom{m}{k}\binom{n}{k} H_{m-k}\left(\frac{y}{(\lambda^2-a)^{\frac{1}{2}}}\right) H_{n-k}\left(\frac{z}{(\mu^2-a)^{\frac{1}{2}}}\right).$$

A. Erdélyi (Edinburgh).

Varma, R. S. Some infinite integrals involving parabolic cylinder functions. Proc. Benares Math. Soc. 1, 61–67 (1939). [MF 1532]

Starting from the equation

$$D_{m-1}(y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp\left(\frac{y^2}{4}\right) \times \int_0^\infty \exp\left(\frac{-u^2}{2}\right) u^{m-1} \sin\left(\frac{m\pi}{2} - uy\right) du,$$

where $D_{m-1}(y)$ is a parabolic cylinder function, the author evaluates the integral

$$\int_0^\infty y^{m-1} \exp(-y^2/4)(y^2+z^2)^{-1} D_{m-1}(y) dy$$

by reversing the order of integration. In the special case that $0<m<2$ an integral equation for the parabolic cylinder function results, using an infinite integral expression of Whittaker for parabolic cylinder functions. Starting from the expression

$$\int_0^\infty \phi(x) \cos mx dx = \exp(-am) \frac{\pi\Gamma(2a)\{\Gamma(b)\}^2}{\{\Gamma(a)\}^2(b+a)\Gamma(b-a)} \cdot {}_2F_1,$$

where

$$\phi = \left(\frac{1+x^2/b^2}{1+x^2/a^2}\right)\left(\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\right)\left(\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\right)\cdots$$

and F_1 is a hypergeometric function, and using Adamoff's integral for parabolic cylinder functions, the author evaluates the integral

$$\int_0^\infty \exp(-x^2/4)\phi(x) D_n(x) dx$$

and a similar more complicated integral. Having previously shown that the functions $x^{\nu-\frac{1}{2}} \exp(x^2/4) D_{-2\nu}(x)$ and $x^{\nu+\frac{1}{2}} \exp(x^2/4) D_{-2\nu-2}(x)$ are self-reciprocal, the self-reciprocity of parabolic functions, the arguments of which are given by the latter expressions, is investigated. Finally, the self-reciprocity of Sonine's polynomials, being generalizations of the parabolic functions dealt with, is investigated by similar methods, evaluating some further infinite integral expressions. *M. J. O. Strutt* (Eindhoven).

Shanker, Hari. On integral representation of Weber's parabolic cylinder function and its expansion into an infinite series. J. Indian Math. Soc. (N.S.) 4, 34–38 (1940). [MF 2033]

Derivation of the integral

$$D_n(z) = -\frac{\Gamma(n-m+1)}{2\pi i} \exp(\tfrac{1}{4}z^2) \times \int_\infty^{(0+)} \exp[-\tfrac{1}{4}(z+t)^2] D_m(z+t)(-t)^{m-n-1} dt,$$

where m and n are arbitrary. The author thence derives the expansion, for arbitrary m,

$$e^{-\frac{1}{4}z^2} D_{n+m}(z) = \Gamma(m+1)\Gamma(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r D_{n-r}(z) D_{m-r}(z)}{\Gamma(n-r+1)\Gamma(m-r+1)r!},$$

which he had previously obtained only for integral values of m. *M. C. Gray* (New York, N. Y.).

Dhar, S. C. Note on the addition theorem of parabolic cylinder functions. J. Indian Math. Soc. (N.S.) 4, 29–30 (1940). [MF 2031]

Extension to non-integral values of m and n of addition theorems for $D_n(x)$ and $D_n(x)D_m(X)$ previously published by the author [J. London Math. Soc. 10, 171–175 (1935)]. The only change in the results is that finite sums are replaced by infinite series. *M. C. Gray.*

Shabde, N. G. On some integrals involving associated Legendre functions. Bull. Calcutta Math. Soc. 31, 87–90 (1939). [MF 2229]

It is shown that, if $f(z) = P_p^m(z)P_q^m(z)$, then for $p \geqq q+2m$, $q \geqq m$

$$\int_1^\infty (z^2-1)^m z^{-l} f(z) dz = 2^{-m} M \sum_{r=0}^{N} BC,$$
$$\Re(l+p-q-2m)>0; \Re(l-p-q-2m)>-1,$$
$$\int_{-1}^{1} (1-u^2)^{k-1} f(u) du = \pi M 2^{-2m} \sum_{r=0}^{N} B_r D_r, \quad \Re(1-m)>0,$$
$$\int_0^\infty e^{-xt} \operatorname{sh}^{2m} t f(\operatorname{ch} t) dt = L \sum_{r=0}^{N} B_r E_r, \quad \Re(x-m)>p+q+m,$$
$$\int_0^\infty e^{-xt} \operatorname{sh}^{2m+1} t f(\operatorname{ch} t) dt = \tfrac{1}{2} Lx\Gamma(m+\tfrac{3}{2}) \sum_{r=0}^{M} B_r F_r,$$
$$\Re(x-m)>p+q+m+1,$$

where $2^{2m+1}L\pi^{\frac{1}{2}} = M = (p+m)!(q+m)!/(p-m)!(q-m)!$, $N = q+m$, $2k=1+m$, $p+q-2r=s$, $m+r+s=v$, $m+s=w$, $x-s=2y$,

$$\Gamma(s+1)(2v+1)B_r A_v^{-m} = (2w+1)\Gamma(w+m+1)A_{r,m}A_{p-r}^m A_{q-r}^m,$$

$(m+n)!A_n^m = (\frac{1}{2})_n$, $r!A_{r,m} = (\frac{1}{2}-m)_r$, $(\frac{1}{2})_n(m-n)!A_{-n}^m = (-1)^n$, $\Gamma(\frac{1}{2})\Gamma(l)C_r = 2^{l-m-2}\Gamma(h)\Gamma(h-w-\frac{1}{2})$, $l+s=2h$,

$$\Gamma(k+\tfrac{1}{2}s+\tfrac{1}{2})\Gamma(k-m-\tfrac{1}{2}s)\Gamma(m+1+\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)D_r = \Gamma(k)\Gamma(k-m),$$
$$\Gamma(y+\tfrac{1}{2})\Gamma(y+w+1)E_r = \Gamma(y+s+\tfrac{1}{2})\Gamma(y-m),$$
$$\Gamma(y+1)\Gamma(y+w+\tfrac{3}{2})F_r = \Gamma(y+s)\Gamma(y-m-\tfrac{1}{2}).$$

H. Bateman (Pasadena, Calif.).

Bagchi, Haridas. The method of difference equations applied to the summation of certain series involving Legendre and Bessel functions. J. Indian Math. Soc. (N.S.) 4, 13–24 (1940). [MF 2029]

Recurrence formulas involving Legendre functions of the first and second kind, as well as Bessel functions, are used in the known manner in order to derive formulas of the Christoffel type. Generating series for the Legendre functions of both kind are also discussed. *G. Szegö.*

Macrobert, T. M. Some integrals involving Legendre and Bessel functions. Quart. J. Math., Oxford Ser. 11, 95–100 (1940). [MF 2607]

The author evaluates three integrals involving associated

Legendre functions, e.g.

$$\int_0^\infty (\sinh u)^{l+m-1}(x^2\cosh^2 u-1)^{-\frac{1}{2}m}Q_n^m(x\cosh u)du,$$

in terms of hypergeometric functions, and shows that these provide alternative derivations of previously published integrals involving Bessel functions, such as

$$\int_0^\infty K_m(x\lambda)K_n(\lambda)\lambda^{l-1}d\lambda.$$

Further integrals of the form

$$\int_1^\infty e^{-x\lambda}(\lambda-1)^{l-1}(\lambda^2-1)^{\pm\frac{1}{2}m}P_n^{-m}(\lambda)d\lambda$$

are also evaluated in terms of E-functions and the results again applied to obtain similar Bessel function integrals.

M. C. Gray (New York, N. Y.).

Banerjee, D. P. On some new integral relations between Bessel and Legendre functions of unrestricted degree. J. Indian Math. Soc. (N.S.) 4, 25–28 (1940). [MF 2030]

Elementary derivation of the integral

$$\int_0^1 P_{n+\frac{1}{2}}(x)J_n(2px)x^{-\frac{1}{2}}dx = \frac{2^{\frac{1}{2}}}{2n+1}[J_n(p)\cos p + J_{n+1}(p)\sin p],$$

where $\Re(n) > -\frac{1}{2}$. The author also derives expansions in series of Bessel functions of integrals involving two Legendre functions, for example,

$$\int_0^1 P_p(1-2y^2)P_q(1-2y^2)J_0(yz)ydy,$$

but in these expansions p and q are restricted to integral values.

M. C. Gray (New York, N. Y.).

Newsom, C. V. and Franck, A. Upon the asymptotic representation of functions of the Bessel type. Bol. Mat. 13, 11–14 (1940). [MF 1726]

Comme application d'un théorème de C. V. Newsom [Amer. J. Math. 60, 561–572 (1938)], les auteurs obtiennent pour la fonction

$$\sum_{n=0}^\infty \frac{z^{2n}}{\Gamma(n+\alpha)\Gamma(n+\beta)}$$

un développement asymptotique qui se réduit pour $\alpha=1$ au développement classique de Hankel.

J. Coulomb.

Brixy, Eduard. Logarithmus der Besselschen Funktionen reellen positiven Argumentes. Z. Angew. Math. Mech. 19, 372–379 (1939). [MF 1473]

The author starts from the formula for the expression $x^n/J_n(x)$, where J_n is Bessel's function of the first kind of order n. After showing that the power series for Bessel's function in the denominator converges uniformly and absolutely, the terms of this series are combined in a new way and this new arrangement is subsequently used for obtaining a new power series for the expression written down above. The coefficients of this new series are determined by an infinity of recurrent linear equations. This system of equations is solved and a simple check is applied to the solution. Hereafter the author considers the function $x^nJ_{n-1}/J_n(x)$. A similar treatment is applied to this expression and a new power series derived for it, the coefficients of which are equally determined by an infinite number of recurrent linear equations. This system is also solved, the solution checked and applied to the case of J_2/J_1. The author finally considers the expression $\log(J_n/x^n)$. This is written as the integral of the expression $xJ_n'(x)/J_n(x)$. A new power series is derived for the latter expression and for its integral, the coefficients of which are determined by making use of the previous developments in this paper. A numerical table of the said coefficients is added.

M. J. O. Strutt (Eindhoven).

Badellino, Maria. Sul calcolo delle funzioni di Bessel. Rend. Sem. Mat. Roma 3, 271–279 (1939). [MF 1906]

Verfasser erwähnt, dass die bisher existierenden Tafeln, welche er aufzählt, der Besselschen Funktionen erster Art sowie der Hankelschen Funktionen der Ordnungen 0 und 1 nur bis zu Argumentwerten von 20 gehen. Für gewisse Anwendungen sind Tafeln dieser Funktionen für Argumentwerte zwischen 20 und 50 notwendig. Nach einem einleitenden Abschnitt über die Definitionen dieser Funktionen, sowie ihrer Potenzreihen und ihrer asymptotischen halbkonvergenten Reihen geht Verfasser dazu über, auf Grund der bekannten Integraldarstellung der Hankelschen Funktionen Ausdrücke für die bei Anwendung der genannten halbkonvergenten Reihen gemachten Höchstfehler abzuleiten. Er gibt an, wie die beigefügte Tafel berechnet ist und welche Kontrollrechnung durchgeführt wurde. Es folgt eine Tafel der genannten Funktionen für Argumente zwischen 20 und 50, mit einem Argumentschritt von 1.

M. J. O. Strutt (Eindhoven).

***Heatley, A. H. Some integrals, differential equations, and series related to the modified Bessel function of the first kind.** University of Toronto Studies, Mathematical Series, no. 7. University of Toronto Press, Toronto, 1939. 32 pp. $1.00.

Part I deals with

$$T(m,n)=\int_0^\infty t^{m-n}e^{-p^2t^2}I_n(2at)dt,$$

and with the related functions $e^{-a^2/p^2}T(m,n)$ and $e^{-a^2/2p^2}T(m,n)$. Differential equations, recurrence formulas and series expansions for these functions are developed in great detail, and values of $T(m,n)$ for particular pairs of values of m and n are obtained. As far as the reviewer can see these results are not new; in fact, for any values of the parameters, $T(m,n)$ can be readily evaluated in terms of a confluent hypergeometric function:

$$T(m,n)=\frac{e^{a^2/2p^2}\Gamma(\frac{1}{2}m+\frac{1}{2})}{2ap^{m-n}\Gamma(n+1)}M_{\frac{1}{2}n-\frac{1}{2}m,\frac{1}{2}n}\left(\frac{a^2}{p^2}\right),$$

and the formulas of this section follow immediately from known properties of the function $M_{k,m}(x)$. In part II the author applies the results of Part I to the numerical evaluation of a complicated triple integral previously discussed by him in an article on the theory of collectors for gaseous discharges. Part III has no apparent connection with the rest of the paper and includes only a brief and incomplete discussion of the evaluation of the indefinite integral

$$\int x^{m+n}e^{\pm x}I_n(x)dx.$$

M. C. Gray (New York, N. Y.).

Meijer, C. S. Ueber Besselsche, Struvesche und Lommelsche Funktionen. II. Nederl. Akad. Wetensch., Proc. **43**, 366–378 (1940). [MF 1913]

The first communication app‸ared in the same Proc. **93**, 198–210 (1940); cf. these Rev. **1**, 233. The present paper contains a considerable number of particular cases of the theorems enunciated in the first part. Thus integral formulae and integral representations are obtained for Bessel, Struve and Lommel functions and for certain products and other combinations of such functions. *A. Erdélyi.*

Stanaitis, O. E. Über die Integralgleichungen der Laméschen und verwandten Funktionen. Mem. Fac. Sci. Univ. Vytautas **13**, 5–46 (1939). (German. Lithuanian summary) [MF 1851]

Verfasser betrachtet im Anschluss an die grundlegenden Arbeiten von E. L. Ince [Proc. Roy. Soc. Edinburgh **42**, 43–53 (1920)] und v. Koppenfels [Math. Ann. **112**, 24–51 (1935)] zunächst allgemein den Zusammenhang des Eigenwertproblems einer linearen homogenen Differentialgleichung zweiter Ordnung mit periodischen Randbedingungen mit der entsprechenden homogenen Integralgleichung und leitet die Regeln ab, welche zur Gewinnung von Kernen solcher Integralgleichungen führen können. Diese Regeln wendet er auf die Differentialgleichung der Laplaceschen Kugelfunktionen an. Diese Differentialgleichung wird mit Hilfe bekannter Transformationsformeln in die Differentialgleichung der Laméschen Potentialfunktionen [vgl. Ergebnisse der Math., Bd. I, Heft 3] transformiert. Die wichtigsten hierbei auftretenden Formeln werden übersichtlich zusammengestellt. Durch Anwendung der zuerst angegebenen allgemeinen Regeln zur Gewinnung von Kernen der dem betrachteten Problem entsprechenden homogenen linearen Integralgleichung kommt Verfasser zu einer ganzen Reihe solcher Kerne für Integralgleichungen der Laméschen Potentialfunktionen, wobei die Jacobischen elliptischen Funktionen zugrunde gelegt sind. Durch Diskussion dieser Integralgleichungen gelingt es, die bekannten Sätze über die Existenz der verschiedenen Klassen Laméscher Potentialfunktionen sowie über ihre Anzahl zu beweisen. Hierauf wendet Verfasser sich den Transformationsformeln zu, welche zu den Differentialgleichungen derselben Laméschen Funktionen unter Zugrundelegung der Weierstrassschen elliptischen Funktionen führen und leitet auch in dieser Form eine Reihe von Kernen der entsprechenden Integralgleichungen für die Laméschen Funktionen her. Durch bekannte Grenzübergänge gewinnt er aus den Differentialgleichungen für die Laméschen Potentialfunktionen jene für die Mathieuschen Funktionen. Diese Grenzübergänge erlauben unmittelbar die Herleitung von Kernen der zu den Mathieuschen Funktionen gehörenden Integralgleichungen aus den für die Laméschen Funktionen gewonnenen Kernen. Durch einen direkten Lösungsansatz, der von der zweidimensionalen Wellengleichung ausgeht, gewinnt Verfasser aufs Neue eine Reihe von Kernen für die Integralgleichung der Mathieuschen Funktionen und zeigt, welche Klassen dieser Funktionen zu den verschiedenen Kernen gehören.

M. J. O. Strutt (Eindhoven).

Ince, E. L. The periodic Lamé functions. Proc. Roy. Soc. Edinburgh **60**, 47–63 (1940). [MF 2178]

Any solution of

$$y''=[n(n+1)k^2\,\mathrm{sn}^2\,u-h]y$$

may be termed a Lamé function, but the present study is restricted to the solutions of period $2K$ or $4K$, where $k^2\leqq 1$ and $n>-\frac{1}{2}$. If $s=\mathrm{sn}\,u$, $c=\mathrm{cn}\,u$, $d=\mathrm{dn}\,u$, there are four distinct types of these solutions which may be represented by the series

$$A_0+\sum A_{2r}s^{2r}\equiv d[C_0+\sum C_{2r}s^{2r}]=Ec_n^{2m}(u,k^2),$$
$$c[A_0'+\sum A_{2r}'s^{2r}]\equiv cd[C_0'+\sum C_{2r}'s^{2r}]=Ec_n^{2m+1}(u,k^2),$$
$$\sum B_{2r+1}s^{2r+1}\equiv d\sum D_{2r+1}s^{2r+1}=Es_n^{2m+1}(u,k^2),$$
$$c\sum B_{2r+1}'s^{2r+1}\equiv cd\sum D_{2r+1}s^{2r+1}=Es_n^{2m+2}(u,k^2),$$

respectively. Tables are given of the characteristic values of h to four decimal places for $k=0.1, 0.5, 0.9$ and $n=-\frac{1}{2}, 0(1)25$. If $\mu=k\,\mathrm{sn}\,u\,\mathrm{sn}\,v$, it is shown that there are 4 pairs of integral equations for $Ec_n^{2m}u$, $Es_n^{2m+1}u$, $Ec_n^{2m+1}u$, $Es_n^{2m+2}u$, in which the kernels are (E_n, dDO_n'), (O_n, dDE_n'), $(cCO_n', cdCDE_n'')$ and $(cCE_n', cdCDO_n'')$, respectively, where $C=\mathrm{cn}\,v$, $D=\mathrm{dn}\,v$, $E_n(\mu)=\frac{1}{2}[P_n(\mu)+P_n(-\mu)]$, $O_n(\mu)=\frac{1}{2}[P_n(\mu)-P_n(-\mu)]$.

H. Bateman (Pasadena, Calif.).

Ince, E. L. Further investigations into the periodic Lamé functions. Proc. Roy. Soc. Edinburgh **60**, 83–99 (1940). [MF 2180]

Following up a suggestion made by Hermite in 1880, Ince shows that all Lamé functions of u of real periods $2K$ or $4K$ can be expressed as trigonometrical series of argument $\mathrm{am}\,u=v$ and that the series terminate in precisely those cases in which the corresponding power series in $\mathrm{sn}\,u$ terminate, when the order n is a positive integer. The basic differential equation is first transformed by the substitutions $v=\mathrm{am}\,u$, $y=w\,\mathrm{dn}\,u$. For the functions of period π in v use is made of the series

$$Ec_n^{2m}(u)=A_0+\sum A_{2r}\cos(2rv)=\mathrm{dn}\,u[C_0+\sum C_{2r}\cos(2rv)],$$
$$Es_n^{2m}(u)=\sum B_{2r}\sin(2rv)=\mathrm{dn}\,u\sum D_{2r}\sin(2rv),$$

while for the functions of period 2π use is made of the series

$$Ec_n^{2m+1}(u)=\sum A_{2r+1}\cos(2r+1)v,$$
$$Es_n^{2m+1}(u)=\sum B_{2r+1}\sin(2r+1)v.$$

The characteristic equation for $2h-n(n+1)k^2$ is in each case expressed by means of a continued fraction convergent when $k^2<1$. It is next shown that there cannot be two independent solutions of period π or 2π in v. The method is similar to that previously used by the author to prove the corresponding theorem for Mathieu functions. The nucleus of an integral equation for one of the periodic Lamé functions is expanded in series of products of the characteristic functions, n being a positive integer. Functions with periods exceeding 2π are finally considered, particularly those of period 4π. Such functions can exist for all values of n and are expressible in a finite form when $2n$ is an odd integer. These finite forms have been considered by Halphen and others, but the results given here are simpler and more significant than those hitherto discovered. The values of h corresponding to these finite forms are found and are tabulated to four decimal places for $k^2=0.1, 0.5, 0.9$, $n=\frac{1}{2}(1)4\frac{1}{2}$.

H. Bateman (Pasadena, Calif.).

Sharma, J. L. On recurrence formulae of the Lamé's functions of the same degree. Proc. Benares Math. Soc. **1**, 69–75 (1939). [MF 1533]

The author starts from the differential equation of Lamé's potential function in the Weierstrassian form [see Ergebnisse der Mathematik, vol. 1, sect. 3, Springer, Berlin, 1932, by Strutt] and defines Lamé's potential functions as poly-

nomials of the usual form. He then considers a two-dimensional differential equation of potential and, transforming it by means of uniformized coordinates, obtains a differential equation which is satisfied by products of the said Lamé's functions of equal order and degree. Observing that the differential quotients of these products with respect to spherical angles are also solutions of the differential equation, the author starts to develop these differential quotients in products of Lamé's functions. Evaluating the coefficients of the resulting series expressions, he shows that these series terminate and hence obtains recurrence expressions for the Lamé functions under consideration. The coefficients of these recurrence expressions are definite integrals of products of Lamé functions and are calculated in detail. It is shown that these coefficients satisfy differential equations of orders higher than two. By adequate splitting of the resulting differential equations the author obtains expressions for the coefficients in terms of Lamé's potential functions.

M. J. O. Strutt (Eindhoven).

Mitra, S. C. On certain expansions involving Whittaker's *M*-Functions. Proc. Edinburgh Math. Soc. (2) **6**, 157–159 (1940). [MF 2848]

The author obtains two expansions for the square of the Whittaker *M*-function in series of *M*-functions:

$$x^{-\frac{1}{2}}M^2_{k,-m}(x)=\sum_{r=0}^{\infty} A_r M_{2k+2r+\frac{1}{2},-2m}(2x),\quad \Re(m)>0,$$

$$x^{-\frac{1}{2}}M^2_{k,-m}(x)=\sum_{r=0}^{\infty} B_r M_{\frac{1}{2}-2m+2r,-2m}(2x),\quad \Re(m)>0,$$

where the coefficients A_r and B_r are determined from the differential equation satisfied by $M^2_{k,m}(x)$. A similar expansion

$$M_{k,m}(x)=\sum_{r=0}^{\infty} C_r x^{\frac{1}{2}m+r+\frac{1}{2}} M_{\frac{1}{2}k,\frac{1}{2}m+r-\frac{1}{2}}(x)\cdot$$

is derived by the same method. *M. C. Gray.*

Horn, J. Über hypergeometrische Funktionen zweier Veränderlichen. Math. Ann. **117**, 384–414 (1940). [MF 3003]

The fully convergent hypergeometric series

$$z=\Gamma_2(a,b,x,y)=\sum(a,n-m)(b,m-n)x^m y^n/(m!n!),\qquad a+b\neq 0,$$

satisfies the set of differential equations

$$z_x=p,\ xp_x=(y-b)z+(a-1-x)p+yq,\ q_x=z;$$
$$z_y=q,\ p_y=z,\ yq_y=(x-a)z+xp+(b-1-y)q$$

with the singular lines of determinateness $x=0$, $y=0$ and the singular lines of undeterminateness $x=\infty$, $y=\infty$. In addition to Γ_2 there are two solutions $z=x^aP(-x,-y)$ and $z=y^bQ(-x,-y)$, where P and Q are simple power series. As the line $x=\infty$ is approached a solution may be expressed in the form of a Laplacian integral $\int_0^\infty w_x(t)\exp(-tx^{\frac{1}{2}})dt$. This corresponds to a factor k^2 of the indicial equation; the solution corresponding to the other factor $k+1$ is next discussed and is expressed in the form

$$z=e^{-x}x^{a+b-1}\Big(\sum_{\nu=0}^{n} f_\nu(y)e^{-y}x^{-\nu}+e_n x^{-n}\Big),$$

where $|e_n|<\epsilon$ if $x>R$ and y lies in a finite interval J. The paper ends with a discussion of solutions in the neighborhood of the point of intersection of the lines $x=\infty$, $y=\infty$, the point being approached along a straight line.

H. Bateman (Pasadena, Calif.).

Horn, J. Hypergeometrische Funktionen zweier Veränderlichen im Schnittpunkt dreier Singularitäten. Math. Ann. **117**, 579–586 (1940). [MF 3097]

Some remarks are first made on the region of convergence of the series

$$F_3(a,a',b,b',c,x,y)=\sum(a,m)(a',n)(b,m)(b',n)x^m y^n/[(c,m+n)m!n!],$$

which was found in a previous paper [Math. Ann. **115**, 435–455 (1938)] to be defined by the inequalities $|x|<1$, $|y|<1$, $xy+x+y>0$, $xy-x-y>0$. Series are next found which correspond to a double root of a determinant equation connected with the differential equations used. Solutions are found in the form of series such as

$$z=u\sum_{n=1}^{\infty} f_n(x,y)=v\sum_{n=1}^{\infty} g_n(x,y)+w\sum_{n=3}^{\infty} h_n(x,y),$$

where the functions u, v, w are defined by means of certain total differential equations and f_n, g_n, h_n are homogeneous functions of the nth degree in the variables x, y. It is finally shown that the system of differential equations for F_3 is satisfied by power series of the form $\xi^\alpha P_1(\xi,-\eta)$, $\xi^\beta P_2(\xi,-\eta)$, where $x\xi=1$, $y=1+\eta$. *H. Bateman* (Pasadena, Calif.).

Cohen, A. C., Jr. The numerical computation of the product of conjugate imaginary gamma functions. Ann. Math. Statistics **11**, 213–218 (1940). [MF 2344]

The difference equation

$$\text{(1)}\qquad \frac{f_{x+1}}{f_x}=\frac{(x-\alpha_1)(x-\alpha_2)}{(x-\beta_1)(x-\beta_2)}$$

has been proposed as the basis for graduating frequency distributions. It has the solution

$$\text{(2)}\qquad f_x=w_x\frac{\Gamma(x-\alpha_1)\Gamma(x-\alpha_2)}{\Gamma(x-\beta_1)\Gamma(x-\beta_2)},$$

where w_x is periodic of period 1. For the case α_1, α_2, β_1, β_2 all real, the gamma functions in (2) can be calculated by means of existing tables. If α_1, α_2 are conjugate complex and/or β_1, β_2 are conjugate complex, some other method must be used. The author shows that: (a) If u is a positive integer,

$$\Gamma(u+iv)\Gamma(u-iv)=\frac{2\pi v}{e^{\pi v}-e^{-\pi v}}\prod_{r=1}^{u-1}(v^2+r^2).$$

(b) If u is not a positive integer,

$$\log\Gamma(u+iv)\Gamma(u-iv)=\log 2\pi+(2u-1)\log R-2(\varphi v+u)+2\psi(R,\varphi),$$

where $u+iv=Re^{i\varphi}$, and

$$\psi(R,\varphi)=\sum_{m=0}^{\infty}\frac{(-1)^m B_{2m+1}}{(2m+1)(2m+2)}\frac{1}{R^{2m+1}}\cos(2m+1)\varphi,$$

in which the B's are the Bernoulli numbers. The author gives numerical examples of the computation and considers the magnitude of the error committed in terminating the series at a given point. *W. E. Milne* (Corvallis, Ore.).

Differential Equations

Jivoïnovitch, P. Remarque sur l'équation de Riccati. Bull. Soc. Math. Grèce **20**, 1–3 (1940). [MF 2630]

New demonstration for some conditions of integrability found by Mitrinovitch [C. R. Acad. Sci. Paris **206**, 411–413 (1938)]. *W. Feller* (Providence, R. I.).

Turrière, E. Sur des courbes spéciales définies par des équations différentielles non intégrables. Enseignement Math. **38**, 69–91 (1940). [MF 2251]

The author considers various special problems in geometry and mechanics which lead in their solution to equations of the form

$$(1)\qquad \frac{dy}{dx}+a_0(x)y^3+a_1(x)y^2+a_2(x)y+a_3(x)=0.$$

The explicit solution of this equation is of course impossible in the general case. In several cases which arise in connection with the problems which the author considers, he is able, by introducing conditions to be satisfied by the functions $a_i(x)$, to find explicit solutions, or to reduce the equation to a well-known form whose solutions have been studied, such as Bessel's equation and special forms of Ricatti's equation. As an example of the type of problem treated, consider the following geometric problem: Let ψ be the angle between the radius vector of a curve in polar coordinates, $\tan\psi=rd\theta/dr$. It is desired to find a curve such that the ordinate, $y=r\cdot\sin\theta$, and ψ have a given relationship:

$$(2)\qquad y=f(\tan\psi).$$

By the introduction of suitable new coordinates, the equation (2) can be reduced to a form where it may be solved explicitly, provided conditions are imposed on f. The conditions appear in the form of a differential equation to be satisfied by f. Several other similar problems are considered in the paper, in connection with pedal curves, curves of pursuit, and the like. *J. W. Green* (Rochester, N. Y.).

Fayet, J. Sur la réduction de certaines équations différentielles non linéaires à des équations à coefficients constants. Bull. Sci. Math. **64**, 38–45 (1940). [MF 2009]

Une expression différentielle $\phi(z)$ d'ordre n et du second degré par rapport à une fonction $z(t)$ et à ses dérivées, à coefficients constants, est caractérisée par satisfaire à la relation

$$\phi\left(z+\frac{dz}{dt}\right)-\phi\left(z-\frac{dz}{dt}\right)\equiv 2\frac{d}{dt}\phi(z).$$

De cette observation l'auteur déduit la condition pour qu'une expression analogue $F(y)$ relative à une fonction $y(x)$ et à ses dérivées, à coefficients fonctions de la variable x, puisse se transformer en une expression à coefficients constants par une transformation de la forme $y=\lambda(x)z+\pi(x)$, $dt=u(x)dx$. Cette condition est qu'il existe cinq fonctions $G(x)$, $g(x)$, $\mu(x)$, $\nu(x)$, $\rho(x)$ par lesquelles résulte vérifiée l'identité

$$G\{F[y+(\mu y'+\nu y+\rho)]-F[y-(\mu y'+\nu y+\rho)]\}=2\frac{d}{dx}[g(x)F(y)].$$

Si on considère l'équation différentielle $F(y)=0$, la condition pour qu'elle soit réductible à une $\phi(z)=0$ à coefficients constants par ladite transformation peut s'exprimer en disant qu'il existe trois fonctions μ, ν, ρ telles que toute solution de l'équation donnée soit solution de

$$F[y+(\mu y'+\nu y+\rho)]-F[y-(\mu y'+\nu y+\rho)]=0.$$

L'auteur avait déjà donné des propositions analogues pour l'équation linéaire et homogène d'ordre n [C. R. Acad. Sci. Paris **204**, 650–652 (1937)]. Sans développer les calculs il donne de même la condition pour qu'une équation du troisième degré, homogène, $F_1(y)=0$ soit réductible à une équation $\phi_1(z)=0$, à coefficients constants, par une transformation de la forme $y=\lambda(x)z$, $dt=u(x)dx$. Il doit exister quatre fonctions G, g, μ, ν de x, telles qu'on ait l'identité

$$G(x)\{F_1[y+(\mu y'+\nu y)]-F_1[y-(\mu y'+\nu y)]-2F_1(\mu y'+\nu y)\}=2\frac{d}{dx}[g(x)F_1(y)],$$

ou bien deux fonctions μ, ν telles que toute solution de $F_1(y)=0$ le soit aussi de

$$F_1[y+(\mu y'+\nu y)]-F_1[y-(\mu y'+\nu y)]-2F_1(\mu y'+\nu y)=0.$$

B. Levi (Rosario).

Mambriani, Antonio. Genesi ed integrazione in termini finiti di vaste classi d'equazioni differenziali lineari, aventi per coefficienti delle funzioni razionali intere. Ann. Scuola Norm. Super. Pisa (2) **9**, 27–43 (1940). [MF 1751]

The author has developed an expansion for the result of applying a linear operator to a product, similar to Leibniz's rule. By an argument largely formal and based on this expansion he constructs certain classes of linear differential equations with rational coefficients which are solvable in finite form. His results include that of de la Vallée Poussin on Bessel's functions. *P. Franklin* (Cambridge, Mass.).

Chiellini, Armando. Sulle pseudo-equazioni differenziali di Fuchs di prima specie, di ordine qualunque e su classi di equazioni di Riccati riducibili alle quadrature. Rend. Sem. Fac. Sci. Univ. Cagliari **9**, 142–155 (1939). [MF 1910]

The author determines a class of linear homogeneous differential equations which behave, with respect to their integration, like an equation of Fuchs of the first kind. By applying his results to the special case of second order equations, he presents a class of Riccati equations which are solvable by quadratures. *W. T. Reid* (Chicago, Ill.).

Chiellini, Armando. Sulle condizioni necessarie e sufficienti affinchè un'equazione differenziale lineare ed omogenea coincida con la propria aggiunta e sopra altre proprietà di tali equazioni di ordine pari. Rend. Sem. Fac. Sci. Univ. Cagliari **9**, 204–214 (1939). [MF 1909]

Grünberg, G. A. Über einige Theoreme der Störungstheorie und insbesondere über deren Anwendung in der Theorie der nichtstationären Erscheinungen in Elektronenröhren. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 22–25 (1939). [MF 2063]

The author considers the differential equation

$$(1)\qquad \ddot{\xi}=f(\xi,\dot{\xi})+\epsilon\varphi(\xi,\dot{\xi},t)$$

and expands the solutions in powers of the parameter ϵ. It is shown that, if the case $\epsilon=0$ is soluble, the various linear equations obtained for the coefficients of the different positive powers of ϵ are soluble by quadratures. Similar results are shown to apply for the case where f involves t also, provided that a one-parameter group of transformations is known which leaves the differential equation $\ddot{\xi}=f(\xi,\dot{\xi},t)$ invariant. Several applications of technical interest are given. *H. Poritsky* (Schenectady, N. Y.).

Sansone, G. Sul comportamento asintotico degli integrali dell'equazione $v'+1+u^n/v=0$, $n>1$. Boll. Un. Mat. Ital. (2) **2**, 105–106 (1940). [MF 2966]

This note contains a direct proof of a result for the first-

order differential equation $v'+1+u^n/v=0$, $n>1$, which in the special case of n a rational value has previously been established by E. Hopf [Monthly Not. Roy. Astr. Soc. **91**, 653–663 (1930)]. The author's method of proof also enables him to obtain the desired result without the aid of a theorem due to Hardy which Hopf employed in the special case of n rational. *W. T. Reid* (Chicago, Ill.).

Bautin, N. Du nombre de cycles limites naissant en cas de variation des coefficients d'un état d'équilibre du type foyer ou centre. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 669–672 (1939). [MF 2058]

It is proved that the coefficients of the system,

$$\frac{dx}{dt}=\sum_{i+k=1}^{2} a_{ik}x^iy^k,\quad \frac{dy}{dt}=\sum_{i+k=1}^{2} b_{ik}x^iy^k,\quad i, k=0, 1, \text{ or } 2,$$

can be so chosen that it may have either 1, 2 or 3 isolated closed solutions in an assigned neighborhood of the solution $x=y=0$, but that the system can not have more than three such solutions. *D. C. Lewis* (Durham, N. H.).

Pipes, Louis A. The analysis of symmetrical vibrating systems. J. Appl. Phys. **11**, 279–282 (1940). [MF 1675]

For a linear system which is invariant under the full symmetric group of permutations of its n degrees of freedom, the matric ("impedance") equation, equivalent to its Lagrangian form by Laplace-transformation, is reduced to its canonical diagonal form which contains two different elements that are linear combinations of the two elements of the original symmetrical matrix. This gives immediately the general solution of the Lagrangian system; application is made to a simple mechanical example of $n=2$.

H. G. Baerwald (Cleveland, Ohio).

Hukuhara, Masuo. Intégration formelle d'un système d'équations différentielles non linéaires dans le voisinage d'un point singulier. Ann. Mat. Pura Appl. (4) **19**, 35–44 (1940). [MF 3043]

The author obtains formal solutions of the formal system

$$x^{\sigma+1}\frac{dy_j}{dx}=\sum a_{j,k_0,k_1,\cdots,k_n}x^{k_0}y_1^{k_1}\cdots y_n^{k_n},$$

$j=1,\cdots,n$; $k_\nu\geqq 0$, $k_1+\cdots+k_n>0$; these solutions contain a number of arbitrary constants and lead without difficulty to those of W. J. Trjitzinsky [Mémor. Sci. Math., v. 90, 1938; Trans. Amer. Math. Soc. **42**, 225–321 (1937)]. The method is that of reductions to canonical systems.

W. J. Trjitzinsky (Princeton, N. J.).

✶Lahaye, Edm. Les itérations intégrales convergentes. Application aux équations différentielles du premier ordre algébriques en y et dy/dx. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. **18**, no. 5, 65 pp. (1939).

The Cauchy-Lipschitz and Picard method of successive approximations for study of solutions of differential equations of the type $y'=f(x, y)$ are quite general and complete in many respects. However, when the path of integration contains "critical points" (that is, points where either $y(x)$ or $f(x, y(x))$ fail to be holomorphic), neither of these methods lends itself for use in a study of the solution in a neighborhood of such points. The present paper considers two cases: one where $f(x, y)$ is the quotient of two polynomials in y, and the other is of the form $F(x, y, y')=0$, where F is a polynomial in y and y'. Methods are developed which for these two cases enable one to study the solutions in the neighborhoods of critical points of a type called mobile (such critical points are not fixed in position but depend upon the initial values assigned to the solution). The author states that he has already published a treatment of cases where the critical points are of fixed type. The methods developed in the paper consist of defining sequences of functions through a process of iteration. These sequences are shown to converge and to represent solutions of the differential equations. Although the approaches used have many elements of similarity to those used in the Cauchy-Lipschitz and Picard processes, the sequences of functions are quite different. This method gives the solutions on more extensive domains than are given by the other processes although the types of equations treated are more limited.

W. M. Whyburn (Los Angeles, Calif.).

Lahaye, Edmond. Les itérations intégrales convergentes et leur application aux equations différentielles du premier ordre, algébriques en y et y'. C. R. Acad. Sci. Paris **210**, 621–624 (1940). [MF 3036]

This note reports the results established by its author in the article reviewed above. The note indicates the method of attack used in the study but fails to give references to the work cited above. *W. M. Whyburn.*

Giuliano, Landolino. Sull'unicità delle soluzioni dei sistemi di equazioni differenziali ordinarie. Boll. Un. Mat. Ital. (2) **2**, 221–227 (1940). [MF 2976]

Given a system of equations $y_i'=f_i(x, y_1, \cdots, y_n)$ $(i=1, \cdots, n)$, where the f_i are defined on a set

$$(*)\qquad \bar{x}<x\leqq\bar{x}+\alpha,\quad -\beta+\bar{y}_i\leqq y_i\leqq\beta+\bar{y}_i,$$

two solutions $y(x)$, $Y(x)$ $(\bar{x}\leqq x\leqq\bar{x}+\alpha)$ of the equations which have the common value $\bar{y}$ at $\bar{x}$ and lie in the set (*) are identical if the following condition holds. There are functions $1/\omega(u)>0$, $M(x)$, defined respectively for $u>0$ and $\bar{x}<x\leqq\bar{x}+\alpha$, having Lebesgue integrals over the respective intervals $[\epsilon, u_0](0<\epsilon<u_0)$ and $[\bar{x}+\epsilon, \bar{x}+\alpha]$ which approach limits respectively ∞ and finite as ϵ tends to 0, and such that

$$(**)\qquad [f_i(x, y_1, \cdots, y_n)-f_i(x, Y_1, \cdots, Y_n)][y_i-Y_i] \leqq M(x)\omega(\sum[y_i-Y_i]^2)$$

whenever $(x, y_1, \cdots, y_n)$ and $(x, Y_1, \cdots, Y_n)$ are distinct points in (*). If the left member in (*) is understood to be summed on i, this generalizes a theorem of the reviewer [Bull. Amer. Math. Soc. **45**, 755–757 (1939); these Rev. **1**, 54]. For $n=1$ it generalizes a theorem of Tonelli [Atti Accad. Naz. Lincei. Rend. **1**, 274–277 (1925)] of which the reviewer had previously been ignorant. *E. J. McShane.*

Giuliano, Landolino. Su un notevole teorema di confronto e su un teorema di unicità per i sistemi di equazioni differenziali ordinarie. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **1**, 330–336 (1940). [MF 2169]

This paper is concerned with solutions of a differential system (1): $y_\nu'=f_\nu(x, y_1, \cdots, y_n)$, $\nu=1, \cdots, n$, on a rectangular region R: $|x-\xi|\leqq\alpha$, $|y_\nu-\eta_\nu|\leqq\beta$. Suppose that $\phi(x, u)$ is a real continuous function on $\xi\leqq x\leqq\xi+\alpha$, $|u|\leqq 2\beta$ and $\phi(x, 0)\geqq 0$ on $(\xi, \xi+\alpha)$; moreover, for a pair of points $P(\xi, \eta_1, \cdots, \eta_n)$, $\bar{P}(\xi, \bar{\eta}_1, \cdots, \bar{\eta}_n)$ of R the maximal solution $u_0(x)$ of $u'=\phi(x, u)$ through the point

$$(\xi, \bar{u}=\max_{\nu=1}^{n}|\eta_\nu-\bar{\eta}_\nu|)$$

satisfies $|u_0(x)|<2\beta$ on $(\xi, \xi+\alpha)$. Whenever f satisfies on R

the inequality

$$(y_\nu - Y_\nu)\{f(x, y_1, \cdots, y_n) - f(x, Y_1, \cdots, Y_n)\}$$
$$\leqq \max_{\nu=1}^{n} |y_\nu - Y_\nu| \phi(x, \max_{\nu=1}^{n} |y_\nu - Y_\nu|), \quad \nu = 1, \cdots, n,$$

the author shows that, if $y_1(x), \cdots, y_n(x)$ and $Y_1(x), \cdots, Y_n(x)$ are solutions of (1) such that $y_\nu(\xi) = \eta_\nu$, $Y_\nu(\xi) = \bar{\eta}_\nu$, $|y_\nu(x) - \eta_\nu| \leqq \beta$, $|Y_\nu(x) - \eta_\nu| \leqq \beta$ on $(\xi, \xi+\alpha)$, then $|y_\nu(x) - Y_\nu(x)| \leqq u_0(x)$, $\nu = 1, \cdots, n$, on this interval. If $P = \bar{P}$, $\phi(x, 0) \equiv 0$ and $u_0(x) \equiv 0$ is the maximal solution of $u' = \phi(x, u)$ through $(\xi, 0)$ on $(\xi, \xi+\alpha)$, it then follows that there exists at most one solution of (1) passing through P and belonging to R. *W. T. Reid* (Chicago, Ill.).

Rellich, Franz. Über die ganzen Lösungen einer gewöhnlichen Differentialgleichung erster Ordnung. Math. Ann. **117**, 587–589 (1940). [MF 3098]

It is proved that a differential equation $dy/dx = f(x, y)$, where $f(x, y)$ is not linear in y, and an entire function of x and y can have at most an enumerable number of solutions $y = w(x)$, where $w(x)$ is an entire function of x. If there are an infinite number of such functions $w_n(x)$, then for every x, as $n \to \infty$, $w_n(x) \to \infty$. *P. Franklin.*

Martin, Monroe H. Real asymptotic solutions of real differential equations. Bull. Amer. Math. Soc. **46**, 475–481 (1940). [MF 2415]

The paper deals with the differential system

$$\frac{dx_k}{dt} = \sum a_{k_j} x_j + \Phi_k(x_1, \cdots, x_n), \quad k = 1, \cdots, n,$$

in which the a_{k_j} denote constants and the Φ_k are power series converging in a neighborhood of the origin $x_1 = \cdots = x_n = 0$, and containing neither constant nor linear terms. The following theorem is established: "If the first m of the characteristic constants λ_k of the matrix $\|a_{k_j}\|$ have negative real parts, while the remaining ones have positive real parts, if the elementary divisors of the characteristic matrix are linear, and if none of the relations

$$\lambda_k = p_1\lambda_1 + \cdots + p_m\lambda_m,$$
$$k = 1, \cdots, m;\ p_k = 0, 1, \cdots;\ p_1 + \cdots + p_m \geqq 2,$$

holds, a point in a suitable neighborhood of the origin lies on a real solution of the system asymptotic to the origin as $t \to +\infty$ if, and only if, it is a point of a certain real, analytic m-dimensional manifold regular at the origin." *R. E. Langer* (Madison, Wis.).

Miller, J. C. P. On a criterion for oscillatory solutions of a linear differential equation of the second order. Proc. Cambridge Philos. Soc. **36**, 283–287 (1940). [MF 2529]

This paper gives a criterion for determining whether real, non-zero solutions of a linear differential equation of the second order have an infinite or finite number of zeros (are oscillatory or non-oscillatory) as the independent variable becomes infinite. The equation is reduced to the form $y'' + R(x)y = 0$ and a set of tests quite similar to the logarithmic tests for convergence of infinite series is developed. The sequence of functions $R(x) = R_0(x)$, $R_1(x) = x^2R_0(x) - 1/4$, $R_2(x) = (\log_e x)^2 R_1(x) - 1/4$, $R_3(x) = (\log_e \log_e x)^2 R_2(x) - 1/4$, $\cdots$ is set up. It is shown that, if $i > 0$ and $\lim_{x\to\infty} R_j(x) = 0$ for $j = i-1$, then (a) all solutions are oscillatory if $R_i(x) > C > 0$ for $x > x_0$; (b) no solution is oscillatory if $R_i(x) \leqq 0$ for $x > x_0$. It is pointed out that these criteria are not exhaustive. *W. M. Whyburn* (Los Angeles, Calif.).

Schmidt, Adam. Konvergente und asymptotische Darstellungen für die Lösungen linearer Differentialgleichungen, deren Koeffizienten Dirichletsche Reihen oder Exponentialpolynome mit komplexen Exponenten sind. Math. Z. **46**, 481–558 (1940). [MF 2781]

The solutions of a linear differential equation

$$\sum_{p=0}^{n} a_p y^{[p]} = 0, \tag{1}$$

or of a differential system

$$y_p' = \sum_{q=1}^{n} a_{p,q} y_q, \qquad p = 1, 2, \cdots, n, \tag{2}$$

with coefficients that are analytic functions of a complex variable z, are in general not single valued in the neighborhood of $z = 0$, if this is a singular point. By the transformation $z = \exp s$ the multiple sheeted neighborhood of $z = 0$ is mapped upon an s-half-plane, the transforms of both (1) or (2) being again of the structure shown. The resulting coefficients, however, appear as Dirichlet series

$$\sum_{\nu=N}^{\infty} d_\nu e^{\lambda_\nu s} \tag{3}$$

with $\lambda_\nu = \nu$. The case $N = 0$ corresponds to that of the regular singular point at $z = 0$, while a negative N is associated with an irregular singular point. These familiar facts motivate the author to regard the differential configurations (1) or (2) with coefficients that are more general Dirichlet series, either infinite, or terminating, that is, exponential polynomials.

Chapter 1 is given to the system (2) with coefficient series in which $N = 0$, the exponents λ_ν are complex,

$$\lambda_0 = 0 < \Re(\lambda_1) \leqq \Re(\lambda_2) \leqq \Re(\lambda_3) \leqq \cdots, \tag{4}$$

and the sequence $\Re(\lambda_\nu)$ has no finite limit. It is shown that when the system is taken in the matrix form $Y' = YA$, there exists a non-singular solution

$$Y = e^{Bs} \sum_{\nu=0}^{\infty} C_\nu e^{\mu_\nu s},$$

in which B is a constant matrix, the exponents μ_ν fulfill relations analogous to those satisfied by the λ_ν and the series converges uniformly in a specifiable region of the s-plane. Chapters 2 and 3 deal with the equation (1) when the exponents in the coefficient series are real and a finite number of them are negative. A specifiable number of Dirichlet series which formally satisfy the equation are deduced, and under stated conditions the existence of an asymptotic solution for s real and $s \to -\infty$ is proved. Finally the restricted form of (1) given by $y'' + ay = 0$ is considered, for the case in which

$$a = e^{2\omega s} \sum_{\nu=0}^{\infty} a_\nu e^{\lambda_\nu s},$$

with ω real and negative, and the λ_ν complex and subject to (4). In this case the existence of an independent pair of asymptotic solutions relative to a half-strip of the s-plane is obtained. *R. E. Langer* (Madison, Wis.).

Cesari, Lamberto. Proprietà asintotiche delle equazioni differenziali lineari ordinarie. Rend. Sem. Mat. Roma **3**, 171–193 (1939). [MF 1901]

It is shown that the results of Perron [Math. Z. **6**, 161–166 (1920); **17**, 149–152 (1923)] on the asymptotic behavior of solutions of ordinary linear differential equations are valid under less stringent conditions than those originally used by Perron. The author also extends his results to the

case of a system of linear differential equations of the first order. *W. T. Reid* (Chicago, Ill.).

Sarymsakoff, T. Sur les lois asymptotiques de la distribution des racines réelles des intégrales oscillatoires d'une équation différentielle linéaire du second ordre. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 322–324 (1939). [MF 2047]

This paper considers the second order, ordinary differential equation: $u''+g(x)u=0$, where $g(x)$ is positive and continuous on an interval (a, b). The double limit

$$h(x)=\lim_{\Delta x\to 0}[\lim_{n\to\infty}\{N_n(x, x+\Delta x)\}/n]/\Delta x,$$

where n is the number of roots of a solution of the differential equation on (a, b), while $N_n(x, x+\Delta x)$ is the number of roots of this same solution on $(x, x+\Delta x)$, is defined to be the asymptotic law of distribution for the roots on (a, b). In the cases where $g(x)$ is either an increasing function or a decreasing function on (a, b), it is shown that the formula

$$h(x)=(g(x))^{\frac{1}{2}}/[\pi n]$$

holds for any solution of the given equation. The proof of the theorem is based directly on a variation of the well-known Sturm separation theorem. Formulas for the polynomials of Tschebycheff, Legendre, Hermite and Laguerre are deduced as special cases of the above.

W. M. Whyburn (Los Angeles, Calif.).

Kowalewski, Gerhard. Bemerkungen über lineare Differentialgleichungen. Deutsche Math. **5**, 116–124 (1940). [MF 2813]

This paper is expository in character and arises from certain lectures given by Sophus Lie. The major part of the paper consists of exhibiting infinitesimal transformations which carry linear nonhomogeneous ordinary differential equations of the second and third orders into themselves. Similar results are given for special equations of order greater than three. *W. M. Whyburn.*

Schin, D. Über die Lösungen einer quasi-Differentialgleichung der n-ten Ordnung. Rec. Math. [Mat. Sbornik] N.S. **7** (49), 479–532 (1940). (Russian. German summary) [MF 2803]

In the background of this work are the lines laid down in some earlier memoirs of H. Weyl [Nachr. Ges. Wiss. Göttingen **1909**, 37–63; Math. Ann. **68**, 220–269 (1909)] relating to certain differential equations of the second order and establishing existence of either one or two distinct solutions $\subset L_2$, depending on whether the case is that of "Grenzpunkt" or "Grenzkreis," the latter being defined by the bilinear form of solutions in the appropriate interval. The author's generalization is for equations

$$(1)\qquad f^{[n]}-lf=0, \qquad \Im(l)\neq 0;\ a<x<b,$$

where

$$f^{[k]}=iP_{k,k}\frac{d}{dx}f^{[k-1]}+\sum_{j=0}^{k-1}P_{k,j}f^{[j]},\quad k=1,\cdots,n;\ f^{[0]}=P_{0,0}f.$$

Under Condition A the P are Lebesgue measurable in (a, b), while the $P_{k,k}$, $P_{k,\nu}$ $(\nu<k;\ k=1,\cdots,n;\ \nu=0,\cdots,n-1)$ $\subset L_2(\alpha,\beta)$ $(a<\alpha<\beta<b)$ for any arbitrary closed sub-interval (α,β). The adjoint $g^{[n]}$ of $f^{[n]}$ is defined in a natural way. The bilinear form for f and g is

$$[f, g]_x=\sum_{k=1}^{n}f^{[n-k]}(x)\bar{g}^{[k-1]}(x).$$

Under a certain Condition B $f^{[n]}$ is self adjoint. If $p^+(\alpha,\beta)[p^-(\alpha,\beta)]$ is to denote the number of distinct solutions of (1), for $\Im(l)>0$ $[\Im(l)<0]$, $\subset L_2(\alpha,\beta)$, then under A, B one has the following four possibilities:

$$p^+(a,c)=n-n',\ p^+(c,b)=n',\ p^-(a,c)=n',\ p^-(c,b)=n-n',\ p^+(a,b)=p^-(a,b)=0;$$
$$p^+(a,c)=n,\ p^+(c,b)=n',\ p^-(a,c)=n,\ p^-(c,b)=n-n',\ p^+(a,b)=n',\ p^-(a,b)=n-n';$$
$$p^+(a,c)=n-n',\ p^+(c,b)=n,\ p^-(a,c)=n',\ p^-(c,b)=n,\ p^+(a,b)=n-n',\ p^-(a,b)=n';$$
$$p^+(a,c)=p^+(c,b)=p^-(a,c)=p^-(c,b)=p^+(a,b)=p^-(a,b)=n.$$

Here $a<c<b$ and n' is the integral part of $n/2$.

In these developments an essential role is played by a system of linear integral equations, which the author solves by successive approximations. *W. J. Trjitzinsky.*

Cinquini, Silvio. Sopra i problemi di valori al contorno per equazioni differenziali del secondo ordine. Ann. Scuola Norm. Super. Pisa (2) **8**, 271–283 (1939). [MF 2041]

Le présent mémoire fait suite à un antérieur [Ann. Scuola Norm. Super. Pisa (2) **8**, 1–22 (1939)]. Par l'application d'un procédé d'approximation polynomiale, on établit deux théorèmes (et quelques corollaires) dans lesquels on donne des conditions suffisantes afin que l'équation $y''=f(x, y, y')$ admette des solutions passant par deux points donnés (x_0, y_0), (x_1, y_1). Les limitations imposées à la fonction $f(x, y, y')$ sont maintenant de la forme $|f(x, y, y')|\leqq\gamma(y)\phi(y')+\psi(x)$; toutefois la énonciation complète de ces conditions ne pourrait être reproduite dans le présent resumé.

B. Levi (Rosario).

Cinquini, Silvio. Problemi di valori al contorno per equazioni differenziali di ordine n. Ann. Scuola Norm. Super. Pisa (2) **9**, 61–77 (1940). [MF 1753]

[Cf. l'analyse précedante.] Il s'agit de théorèmes d'existence, qui assignent des conditions par lesquelles une équation du type $y^{(n)}=f(x, y, y', \cdots, y^{(n-1)})$, ou, comme l'auteur préfère d'écrire pour plus de genéralité,

$$y^{(n-1)}(x)=y^{(n-1)}(x_1)+\int_{x_1}^{x}f(x, y, y', \cdots, y^{(n-1)})dx,$$

admet une solution satisfaisant à n conditions de la forme $y(x_i)=y_i$ $(i=1, 2, \cdots, n)$, ou, en genéral, à n conditions capables de determiner un polynome de degré $n-1$. Les cas considérés sont plus genéraux d'autres résultats analogues obtenus précédemment par Cacciopoli, Scorza-Dragoni et Zwirner. Pour la fonction f l'auteur admet qu'elle soit définie pour toutes les valeurs des $y, y', \cdots, y^{(n-1)}$ et pour les valeurs de x dans un intervalle fini (a, b); et qu'elle soit continue par rapport au groupe des premières variables et quasi-continue par rapport à x et qu'elle vérifie dans tout le champ une inégalité de la forme

$$|f(x, y, y', \cdots, y^{(n-1)})|\leqq\sum\alpha_i(x)|y^{(i)}|+\psi(x),$$

ou

$$|f(x, y, y', \cdots, y^{(n-1)})|\leqq\gamma_1(y^{(n-2)})\varphi(y^{(n-1)})$$
$$+[\gamma_2(y^{(n-3)})|y^{(n-2)}|+\cdots+\gamma_{n-1}(y)|y'|+\psi_1(x)]\varphi_1(y^{(n-1)}),$$

où les fonctions φ, φ_1 ne doivent pas croître trop rapidement (outre à quelques autres conditions de mineure importance pour lesquelles nous renvoyons au mémoire). *B. Levi.*

Kamke, E. Über die definiten selbstadjungierten Eigenwertaufgaben bei gewöhnlichen linearen Differentialgleichungen. II. Math. Z. 46, 231–250 (1940). [MF 2397]

This paper is concerned with a self-adjoint boundary problem of the form $F(y)=\lambda G(y)$, $U_\mu(y)=0$, $\mu=1,\cdots,2m$, where

$$F(y)\equiv\sum_{\nu=0}^{m}[f_\nu(x)y^{(\nu)}]^{(\nu)},\quad G(y)\equiv\sum_{\nu=0}^{n}[g_\nu(x)y^{(\nu)}]^{(\nu)},\quad 0\leqq n<m,$$

and the coefficients f_ν, g_ν are real and of class $C^{(\nu)}$ on $a\leqq x\leqq b$; moreover, $f_m\neq0$, $g_n\neq0$ on this interval. The $U_\mu(y)$ are $2m$ independent linear forms in the end-value of $y, y', \cdots, y^{(2m-1)}$ at a and b with real coefficients. The author discusses systems of this type that are "normal" in the sense that $\int_a^b\bar{\psi}G(\psi)dx\neq0$ for each characteristic solution ψ. In particular, it is shown that for such normal problems the characteristic values are all real, the index of each characteristic value is equal to its multiplicity, and that the Green resolvent has at most poles of the first order.

W. T. Reid (Chicago, Ill.).

Kamke, E. Über die definiten selbstadjungierten Eigenwertaufgaben bei gewöhnlichen linearen Differentialgleichungen. III. Math. Z. 46, 251–286 (1940). [MF 2398]

This paper considers the boundary problem $F(y)=\lambda G(y)$, $U_\mu(y)=0$, $\mu=1,\cdots,2m$, of the preceding paper under the following condition of "definiteness": $\int_a^b uF(u)dx\geqq0$ for each function u which is of class $C^{(2m)}$ on ab and satisfies $U_\mu(u)=0$; moreover, for each such $u\neq0$ rendering $\int_a^b uF(u)dx=0$, the functional $\int_a^b uG(u)dx$ is different from zero and of a fixed sign. The problem is said to be properly definite if the former functional is positive for each such $u\neq0$ on ab. Such a definite system is normal in the sense of the preceding paper; moreover, the system has infinitely many characteristic values. Various properties of the characteristic values are discussed, including their extremizing properties and their approximation by the Ritz-Galerkin and iteration methods.

W. T. Reid (Chicago, Ill.).

Krein, M. et Finkelstein, G. Sur les fonctions de Green complètement non-negatives des opérateurs différentiels ordinaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) 24, 220–223 (1939). [MF 2906]

Let $G(x,s)$ be a Green's function for the differential equation

$$\sum_{k=0}^{m} l_k(x)\frac{d^ky}{dx^k}=0,$$

where $l_k(x)$ $(k=0,\cdots,m)$ is continuous on (a,b): $a\leqq x\leqq b$ and $l_m(x)\neq0$ on (a,b). Let

$$G(x,s)=\sum_{j=1}^{p}u_j(x)v_j(s)$$

for $x\leqq s$,

$$G(x,s)=\sum_{j=1}^{q}u_j^*(x)v_j^*(s)$$

for $x\geqq s$. The function $G(x,s)$ is defined to be completely non-negative if for $n=1,2,\cdots$ the determinant

$$G\begin{pmatrix}x_1 & x_2 & \cdots & x_n\\ s_1 & s_2 & \cdots & s_n\end{pmatrix},$$

whose element in the ith row and jth column is $G(x_i,s_j)$, where $a<x_1<x_2<\cdots<x_n<b$, $a<s_1<s_2<\cdots<s_n<b$; $n=1,2,\cdots$, is non-negative. The present paper establishes the fact that the condition $s_i<x_{i+p}$ $(i=1,2,\cdots,n-p)$, $x_i<s_{i+q}$ $(i=1,2,\cdots,n-q)$ is necessary and sufficient that a completely non-negative $G(x,s)$ have the further property

$$G\begin{pmatrix}x_1 & x_2 & \cdots & x_n\\ s_1 & s_2 & \cdots & s_n\end{pmatrix}>0$$

for $a<x_1<x_2<\cdots<x_n<b$; $a<s_1<s_2<\cdots<s_n<b$; $n=1,2,\cdots$.

W. M. Whyburn (Los Angeles, Calif.).

Krein, M. Sur les fonctions de Green non-symétriques oscillatoires des opérateurs différentiels ordinaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 643–646 (1939). [MF 2119]

The differential operator considered

$$L(y)=\sum_{k=0}^{n} l_k(x)\frac{d^ky}{dx^k},\qquad n\geqq2,$$

is assumed to be self-adjoint with $l_k(x)$ continuous and $l_n(x)>0$ on the closed interval (a,b). Then a Green's function $G(x,s)$ has the form:

$$(1)\qquad G(x,s)\begin{cases}=\sum_{i=1}^{q}\psi_i(x)\chi_i(s), & x\leqq s,\\ =\sum_{i=1}^{p}\psi_i^*(x)\chi_i^*(s), & x\geqq s,\end{cases}$$

where each set $\psi_i(x)$ and $\psi_i^*(x)$ are linearly independent solutions of the homogeneous equation $L(y)=0$ and $p+q\geqq n$. The paper is concerned with the case in which $\epsilon G(x,s)$, $\epsilon=\pm1$, is oscillating, that is, the determinants

$$\epsilon^nG\begin{pmatrix}x_1 & \cdots & x_n\\ s_1 & \cdots & s_n\end{pmatrix}>0$$

for $a<x_1<x_2\cdots<x_n<b$ and $a<s_1<s_2\cdots<s_n<b$ for all n [see Kellogg, Amer. J. Math. 40, 147 (1918)]. It follows that if $\epsilon G(x,s)$ is oscillating then $\epsilon=(-1)^q$ and $p+q=n$. Consequently $G(x,s)$ corresponds to non-singular boundary conditions of the form:

$$(2)\qquad \begin{aligned}&\sum_{k=0}^{n-1}\alpha_{ik}y^{(k)}(a)=0, && i=1,\cdots,p,\\ &\sum_{k=0}^{n-1}\beta_{jk}y^{(k)}(b)=0, && j=1,\cdots,q.\end{aligned}$$

This induces the fact that the conditions $y^{(i)}(a)=0$, $i=0,\cdots,p-1$, $y^{(j)}(b)=0$, $j=0,\cdots,q-1$ are non-singular and the corresponding Green's function $G^{(p,q)}(x,s)$ multiplied by $(-1)^q$ is oscillating. Conversely, if these last two conditions are satisfied, $L(y)$ is expressible in the form

$$\rho_0\frac{d}{dx}\rho_1\frac{d}{dx}\cdots\frac{d}{dx}\rho_ny,$$

where $\rho_k(x)$ are positive functions having k continuous derivatives. Again, if $(-1)^qG(x,s)$ is an oscillating kernel satisfying non-singular conditions (2), then the characteristic numbers of the integral equation

$$\varphi(x)=\lambda\int_a^b G(x,s)\varphi(s)d\sigma(s),$$

$\sigma(s)$ monotonic non-decreasing, are simple zeros of the Fredholm determinant with $(-1)^q\lambda_i$ positive. Order relations exist between the characteristic numbers for a $G(x,s)$ and corresponding $G^{(p,q)}(x,s)$.

T. H. Hildebrandt.

Krein, M. Les théorèmes d'oscillation pour les opérateurs linéaires différentiels d'ordre quelconque. C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 719–722 (1939). [MF 2124]

Assuming a differential operator

$$L(y)=\rho_0\frac{d}{dx}\rho_1\frac{d}{dx}\cdots\frac{d}{dx}\rho_n y,$$

$\rho_k(x)>0$, with continuous kth order derivatives, the paper concerns solutions of $L(y)-\lambda\rho y=0$, ρ Lebesgue integrable and positive almost everywhere, subject to boundary conditions $y^{(i)}(a)=0$, $i=0,1,\cdots,p-1$; $y^{(j)}(b)=0$, $j=0,\cdots,q-1$; $p+q=n$. By the note reviewed above, the characteristic constants are simple and satisfy the conditions $(-1)^q\lambda_i>0$ and can be arranged in order of magnitude. The principal oscillation theorem asserts that, if $W_q(x,\lambda)=W(\omega_1,\cdots,\omega_q)$ is the Wronskian of q linearly independent solutions of the equation $L(y)-\lambda\rho y=0$ subject to the conditions $y^{(i)}(a)=0$, $i=0,\cdots,p-1$, then $W_q(x,\lambda)$ does not vanish in $a\leqq x<b$ if $-\infty<(-1)^q\lambda<(-1)^q\lambda_0$, but has exactly k simple zeros in this interval if $(-1)^q\lambda_{k-1}\leqq(-1)^q\lambda<(-1)^q\lambda_k$. The zeros $x_i(\lambda)$ are decreasing functions of $(-1)^q\lambda$ and

$$\lim_{(-1)^q\lambda\to\infty}x(\lambda)=a.$$

It is announced further that the system $L(y)-\lambda\rho y=0$; $y^{(i)}(a)=0$, $i=0,\cdots,p-1$; $y^{(p)}(a)=1$; $y^{(j)}(b)=0$, $j=0,\cdots,q-2$ has a unique solution $\varphi_q(x,\lambda)$ which has properties similar to those of $W_q(x,\lambda)$. *T. H. Hildebrandt.*

Smogorshewsky, Alexandre. Les fonctions de Green des systèmes différentiels linéaires dans un domaine à une seule dimension. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 179–196 (1940). (French. Russian summary) [MF 2285]

This paper studies one-dimensional Green's functions for linear differential systems consisting either of a single nth order differential equation with boundary conditions or a set of first order equations with boundary conditions. The boundary conditions are expressed by means of Stieltjes integrals. Existence of the Green's functions, generalized Green's functions, Green's matrices and generalized Green's matrices is established for the various possible cases of the systems considered. Somewhat explicit forms are given for the Green's functions and properties of these functions are stated. The special cases where the boundary conditions are of two point type are given detailed treatment. When the coefficients of the differential equations depend on a complex parameter and two point boundary conditions are used, the author develops conditions under which the Green's function (or matrix) is Hermitian in character, that is, $G(x,s)=\pm\bar{G}(s,x)$ or $G_{ij}(x,s)=\pm\bar{G}_{ji}(s,x)$ $(i,j=1,\cdots,n)$. The paper contains an extensive bibliography of papers that concern the one-dimensional Green's function.

W. M. Whyburn (Los Angeles, Calif.).

Galbraith, A. S. and Warschawski, S. E. The convergence of expansions resulting from a self-adjoint boundary problem. Duke Math. J. 6, 318–340 (1940). [MF 2317]

This paper is concerned with a problem of the Riesz-Fischer type having to do with the expansion of a function with n derivatives in terms of the characteristic solutions of a self-adjoint boundary value problem of the second order. The self-adjoint differential system involved consists of the equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]-q(x)y+\lambda r(x)y=0,$$

together with a pair of two-point, linear boundary conditions with constant coefficients. The principal results of the paper consist of establishing: (1) If $\lambda_0,\lambda_1,\cdots$ are the characteristic numbers and $\varphi_0(x),\varphi_1(x),\cdots$ are the characteristic solutions (orthogonal and normal with weight function $r(x)$) of the differential system and if $[a_k]$ is any sequence of real constants such that $\sum_{k=0}^{\infty}a_k^2\lambda_k^n$ converges, then there exists a function $f(x)$ such that

$$f(x)\equiv\sum_{k=0}^{\infty}a_k\varphi_k(x),\quad a_k=\int_a^b fr\varphi_k dx,$$

$f^{(n-1)}(x)$ is absolutely continuous, and the function $[f^{(n)}(x)]^2$ is integrable on (a,b). (2) The first $n-1$ derivatives of the function $f(x)$ considered under (1) are given by the uniformly convergent series obtained by termwise differentiation of the series for $f(x)$. Furthermore

$$\lim_{N\to\infty}\int_a^b[f^{(n)}(x)-\sum_{k=0}^{N}a_k\varphi_k^{(n)}(x)]^2dx=0.$$

(3) A type of converse theorem for (1) in which are given conditions on a function $F(x)$ under which the series $\sum_{k=0}^{\infty}a_k^2\lambda_k^n$ converges, where $a_k=\int_a^bF(x)r(x)\varphi_k(x)dx$. Furthermore, if $F(x)$ meets these conditions, the results described under (2) are valid for $F(x)$. In addition to the main results of the paper, a number of lemmas are established, giving useful information about the differential system and its solutions. *W. M. Whyburn* (Los Angeles, Calif.).

Titchmarsh, E. C. On expansions in eigenfunctions. II and III. Quart. J. Math., Oxford Ser. 11, 129–145 (1940). [MF 2611, 2612]

In a previous paper [J. London Math. Soc. 14, 274–278 (1939); these Rev. 1, 56] the author developed a method of obtaining expansions of an arbitrary function $\psi(x)$ in series of solutions of the differential equation $L\psi=\lambda\psi$, where L is a given differential operator, considering representation by means of the Fourier integral of the solution of the partial differential equation $L\phi=\partial\phi/\partial t$ reducing to $\psi(x)$ for $t=0$. In the present notes the author gives an analogous method which can be applied to more general cases. Let $\phi(x,t)$ be a solution of $L\phi=i\partial\phi/\partial t$, which reduces to $\psi(x)$ for $t=0$ and is $O(e^{A|t|})$ as $t\to\pm\infty$. On setting

$$\Psi_+(x,w)=(2\pi)^{-\frac12}\int_0^{\infty}\phi(x,t)e^{iwt}dt,$$

$$\Psi_-(x,w)=(2\pi)^{-\frac12}\int_{-\infty}^{0}\phi(x,t)e^{iwt}dt,$$

it is readily proved that

$$L\Psi_+=-i(2\pi)^{-\frac12}\psi(x)+w\Psi_+,$$

with an analogous equation for Ψ_-. This determines $\Psi_\pm(x,w)$ and the required expansion is obtained from

$$\phi(x,t)=(2\pi)^{-\frac12}\int_{ia-\infty}^{ia+\infty}\Psi_+(x,w)e^{-iwt}dw+(2\pi)^{-\frac12}\int_{ib-\infty}^{ib+\infty}\Psi_-(x,w)e^{-iwt}dw$$

by using Cauchy's theorem and putting $t=0$. The author shows in several examples how this formal method may be justified. In the third note it is shown how the method above may be used to obtain the Hermite expansion of $\psi(x)$.

J. D. Tamarkin (Providence, R. I.).

Golab, St. Un théorème de la théorie des équations différentielles approchées. Mathematica, Cluj **16**, 61–65 (1940). [MF 2485]

On pose un problème très indeterminé, tel que: remplacer une équation différentielle $dx/dt=f(x)$ par une autre plus simple $dx/dt=f^*(x)$ de telle façon que les intégrales des deux équations qui prennent la même valeur pour $t=0$, tiennent même ordre de grandeur pour $t\to\infty$; problème qui doit évidemment être précisé en limitant convenablement la nature des fonctions $f(x)$, $f^*(x)$. La solution envisagée, sous les hypothèses $f(x)<0$ pour $x>0$, $f(x)\to 0$ pour $x\to 0$, $df(0)/dx=-\lambda<0$, établit que, pour $t>0$, le rapport entre $x(t)$ et $e^{-\lambda t}$ reste compris entre deux nombres positifs (non mieux determinés), au moyen de conditions supplémentaires partiellement excessives et partiellement insuffisantes (on sous-entend une condition, qui pourrait être la continuité de $f(x)$, pour assurer qu'une certaine intégrale résulte finie) et avec des calculs plus labourieux que necessaire.

B. Levi (Rosario).

Ritt, J. F. On the intersections of irreducible components in the manifold of a differential polynomial. Proc. Nat. Acad. Sci. U. S. A. **26**, 354–356 (1940). [MF 2144]

It is known [Ritt, Differential Equations from the Algebraic Standpoint, New York, 1932] that any system Σ of differential polynomials is equivalent to a finite set of irreducible systems Σ_i. If we suppose that the solutions of Σ_i are not all contained in Σ_j, $i\neq j$, then the Σ's are called the irreducible components. If F is a form in $y_1, \cdots, y_n$, a letter y_{ij} is called a letter in F if the order of F in y_i is at least j. The author proves that, if there is a letter y_{ij} in F such that $\partial F/\partial y_{ij}$ is not annulled by a solution $\bar{y}_1, \cdots, \bar{y}_n$ of F, then the solution is contained in only one irreducible component of the manifold of F.

N. Jacobson.

McGavock, William G. Annihilators of quadratic forms with applications to Pfaffian systems. Duke Math. J. **6**, 462–473 (1940). [MF 2330]

An algebra with a basis $u_1^{i_1}, \cdots, u_n^{i_n}$, where $u_i^2=0$, $u_iu_j=-u_ju_i$ over a field $\mathfrak{R}$, is called a Grassmann ring $\mathfrak{G}$ [J. M. Thomas, Differential Systems, New York, 1937]. F in $\mathfrak{G}$ is a form if all of the non-zero monomials have the same degree and S is a pencil of forms if together with 0 it forms a linear subspace of $\mathfrak{G}$. The order of a set of forms is the minimum degree for monomials which annihilate the set. The half rank ρ of a pencil S of quadratic forms is the integer such that there exists a product of ρ forms in S not 0 but no non-zero product of $\rho+1$ forms. The author proves that if F and G are quadratic forms, $\Omega=\omega_1\cdots\omega_r$, ω_i linear such that $\Omega F^\rho\neq 0$, $\Omega G^\rho\neq 0$ but $\Omega F^\lambda G^{\rho-\lambda+1}=0$ for $\lambda=1, \cdots, \rho+1$, then ΩF and ΩG have a common annihilator $u_1\cdots u_\rho$. This is used to show that the order of a pencil of quadratic forms may have any value between ρ and 2ρ and only these. These results are applied to construct Pfaffian systems with species σ [Thomas, Trans. Amer. Math. Soc. **35**, 356–371 (1933)] any integer such that $\rho\leqq\sigma\leqq 2\rho$, ρ the half-rank. If r is the maximum number of linearly independent forms in the system, $\sigma\leqq 2\rho+r-1$.

N. Jacobson.

Schouten, J. A. und van der Kulk, W. Beiträge zur Theorie der Systeme Pfaffscher Gleichungen. II. Beweis des Haupttheorems für $q=n-5$. Nederl. Akad. Wetensch., Proc. **43**, 179–188 (1940). [MF 1591]

[Continuation of a paper in Nederl. Akad. Wetensch., Proc. **43**, 18–31 (1940); these Rev. **1**, 145.] The case of semi-rank 1 is proved with the aid of work of Dearborn [Duke Math. J. **2**, 705–711 (1936)] and Griffin [Trans. Amer. Math. Soc. **35**, 929–939 (1933)]. A corollary of the principal theorem is found, stating that the bilinear system of partial differential equations in n variables

$$\underset{1}{\mu}X_\alpha\overset{1}{p}+\underset{2}{\mu}X_\alpha\overset{2}{p}=0, \qquad \alpha=1, \cdots, p,$$

has then and only then $n-p$ systems of solutions $\overset{1}{p}, \overset{2}{p}$, whose corresponding vectors $\underset{1}{\mu}\partial_\lambda\overset{1}{p}+\underset{2}{\mu}\partial_\lambda\overset{2}{p}$ are linearly independent, if $p\leqq 3$ or the rank of the adjoint system of Pfaff equations is not greater than 2.

D. J. Struik (Cambridge, Mass.).

Schouten, J. A. und van der Kulk, W. Beiträge zur Theorie der Systeme Pfaffscher Gleichungen. III. Beweis des Haupttheorems für den Fall dass der Rang den höchsten Wert hat. Nederl. Akad. Wetensch., Proc. **43**, 453–462 (1940). [MF 2256]

[Cf. the preceding review.] The principal theorem can be outlined as follows. A system of $q=n-p$ linearly independent Pfaff equations in n variables

$$(1) \qquad C_\lambda^x dx^\lambda=0, \quad x=p+1, \cdots, n; \lambda=1, \cdots, n,$$

is given and

$$B_b^\kappa\partial_\kappa f=0, \quad B_b^\mu C_\mu^x=0, \qquad b=1, \cdots, p,$$

is the adjoint system of p linearly independent equations. Let the semi-rank of (1) be equal to ρ. In this case the system of $p-\rho$ differential equations with $\rho+1$ unknowns

$$\underset{1}{\mu}X_\alpha\overset{1}{p}+\cdots+\underset{\rho+1}{\mu}X_\alpha\overset{\rho+1}{p}=0, \quad \alpha=1, \cdots, p; X_b=B_b^\mu\partial_\mu,$$

has a system of solutions $\overset{1}{p}, \cdots, \overset{\rho+1}{p}$, which satisfies at a given point $x^\kappa=\underset{0}{x^\kappa}$ a particular differential relation and a particular algebraic inequality which are given and explained in the text. The quantity $C_{[\lambda_1}^{p+1}\cdots C_{\lambda_q]}^{n}$ has q linearly independent divisors w_λ, of class not greater than $2\rho+1$, but there are no q linearly independent divisors of class all less than $2\rho+1$.

D. J. Struik (Cambridge, Mass.).

Schouten, J. A. und van der Kulk, W. Beiträge zur Theorie der Systeme Pfaffscher Gleichungen. IV. Beweis des Haupttheorems für den Fall, dass der Rang einen beliebigen Wert hat. Nederl. Akad. Wetensch., Proc. **43**, 674–686 (1940). [MF 3106]

Proof of the principal theorem of the theory of systems of Pfaffian equations for the case of general rank. See the two preceding reviews.

D. J. Struik (Cambridge, Mass.).

Pfeiffer, G. V. Les systèmes jacobiens généralisés d'équations linéaires aux dérivées partielles du premier ordre à plusieurs fonctions inconnues et la méthode spéciale d'intégration. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 251–268 (1939). (French. Russian summary) [MF 2297]

The author has previously published his main ideas [Bull. Acad. Sci. Ukraine **3**, 7–20 (1928)]. In the present paper he gives an enlarged representation from a more general

point of view. If a generalized jacobian system

$$(1)\qquad p_{ij}+\sum_{\lambda=1}^{g} a_\lambda^j p_{i\lambda}=b_i^j,\quad i=1,2,\cdots,k;\ j=g+1,\cdots,n,$$

is connected with a non-complete system

$$(2)\qquad \frac{\partial f}{\partial x_j}+\sum_{i=1}^{k} b_i^j\frac{\partial f}{\partial z_i}+\sum_{\lambda=1}^{g} a_\lambda^j\frac{\partial f}{\partial x_\lambda}=0,$$

and if the system of

$$\frac{\partial f}{\partial x_h}+\sum_{i=1}^{k} r_{ti}\frac{\partial f}{\partial z_i}+\sum_{\mu=1}^{h-1} s_{t\mu}\frac{\partial f}{\partial x^\mu}=0,$$

$$q+1\ [illegible]\ h=g,g-1,\cdots,q+1;\ t=g-h+1,$$

and (2) is equivalen[illegible]e complete system

$$(3)\qquad \frac{\partial f}{\partial x_l}+\sum_{i=1}^{k} \overset{l}{b}_{1i}\frac{\partial f}{\partial z_i}+\sum_{\alpha=1}^{q} \overset{l}{a}_{1\alpha}\frac{\partial f}{\partial x_\alpha}=0,\quad l=q+1,\cdots,n,$$

then by adjunction of the equations

$$p_{ih}+\sum_{\mu=1}^{h-1} s_{t\mu}p_{i\mu}=r_{ti},\qquad i=1,2,\cdots,k,$$

to equations (1), we get the generalized jacobian system

$$p_{il}+\sum_{\alpha=1}^{q} \overset{l}{a}_{1\alpha}p_{i\alpha}=\overset{l}{b}_{1i}.$$

A method is indicated to integrate complete systems in such a way that the complete integral is found.

D. J. Struik (Cambridge, Mass.).

Orloff, Constantin. Sur l'intégrale générale des équations différentielles aux dérivées partielles du second ordre. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **6**, 191–197 (1939). [MF 2670]

After discussing the various definitions of the general integral of a second order partial differential equation, the author proceeds to state conditions under which a general integral according to Lagrange's definition is also a general integral according to Ampère's definition. The reviewer was unable to supply the details in the proof sketched by the author. *F. G. Dressel* (Durham, N. C.).

Saltykow, N. Méthodes immédiates d'intégration d'équations aux dérivées partielles du second ordre. Enseignement Math. **38**, 132–159 (1940). [MF 1879]

This paper summarizes a collection of methods, each of which reduces the integration of a certain class of equations

$$F(x,y,z,p,q,r,s,t)=0$$

to a presumably simpler problem. First various instances are pointed out where the problem reduces to ordinary differential equations if certain of the arguments in F are missing. In other cases the problem reduces to first order partial differential equations. Then the author treats numerous examples from the literature in which devices have been used to obtain solutions depending on arbitrary functions. *E. W. Titt* (Hyattsville, Md.).

Halpern, S. Sur les conditions pour que le problème de Cauchy pour un système compatible d'équations linéaires aux dérivées partielles soit correctement posé. Rec. Math. [Mat. Sbornik] N.S. **7 (49)**, 111–141 (1940). (Russian. French summary) [MF 2282]

This paper extends and applies results of a paper by I. Petrowsky [Über das Cauchy'sche Problem für Systeme von partiellen Differentialgleichungen, Rec. Math. [Mat. Sbornik] N.S. **2 (44)**, 815–868 (1937)]. The author considers a Cauchy problem for a certain system (I) of linear equations in several unknown functions of several variables, with the number of equations a multiple of that of the unknown functions. As a first step, the essentially algebraic, necessary conditions of compatibility are derived. The aim of the paper, then, is to decide when Cauchy's problem is "correctly formulated" in Petrowsky's sense; that is substantially when there exists a solution depending continuously on the initial data and their derivatives up to a sufficiently high order. As the theorems deal with a rather complex situation requiring numerous qualifications, it is impossible to give their precise statements here.

H. Lewy (Berkeley, Calif.).

Chaundy, T. W. Linear partial differential equations. II. Quart. J. Math., Oxford Ser. **11**, 101–110 (1940). [MF 2608]

L'auteur traite d'étendre la méthode de Riemann pour la résolution du problème de Cauchy pour les équations aux dérivées partielles de type hyperbolique au cas d'une équation $\bar{\phi}V=f(xy)$ linéaire, en deux variables d'ordre $m>2$, à caractéristiques distinctes. À cet effet on introduit un système de m fonctions de Green $U_1, U_2, \cdots, U_m$, une pour chaque caractéristique, solutions de l'équation adjointe $\phi U=0$, conditionnées à tenir pour somme 0, à s'annuler avec leurs dérivées jusqu'à l'ordre $m-3$ sur la caractéristique corréspondante passant pour le point (XY) dans lequel on traite de calculer la valeur de V, et à satisfaire sur cette caractéristique à une dernière condition dépendant des termes d'ordre $m-1$ de ϕ et dont l'énonciation résulte un peu plus compliquée. On applique alors la transformation de Gauss aux m intégrales $\iint(V\phi U_i-U_i\bar{\phi}V)dxdy$ étendues respectivement aux aires comprises entre un arc γ joignant le point (XY) à un point A de la courbe sur laquelle on connaît les donnés de Cauchy, cette courbe et l'arc de la caractéristique correspondante passant pour (XY). En sommant la partie correspondante à l'arc γ s'élimine à cause de la première condition, pendant que, à cause particulièrement de la dernière condition, la partie correspondante à chaque caractéristique s'intègre, et permet de isoler $V(XY)$. Le problème de la determination effective des fonctions de Green postulées n'est pas traité.

B. Levi (Rosario).

Soloviev, P. V. Fonctions de Green des équations paraboliques. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 107–109 (1939). [MF 2915]

Let $g(x,t,\xi|b)$ be the Green's function for the equation $u_{xx}-u_t=0$ and the region $0\leqq x\leqq b$, $0\leqq t\leqq h$ [cf. G. Doetsch, Theorie und Anwendung der Laplace-Transformation, Berlin, 1937, p. 358]. The author writes out in detail that the product $g(x,t,\xi|b)g(y,t,\eta|c)$ is the Green's function for the equation $u_{xx}+u_{yy}-u_t=0$ and the region $0\leqq x\leqq b$, $0\leqq y\leqq c$, $0\leqq t\leqq h$. At the end of the paper the problem of finding a Green's function for the region $k_it\leqq x_i\leqq k_i't+a_i$, $i=1,\cdots,n$, is considered; but, partly due to misprints, the function given by the author has none of the properties desired of a Green's function. *F. G. Dressel* (Durham, N. C.).

Kienast, Alfred. Ueber einige Fälle der Green'schen Funktion der Wärmeleitung. Vierteljschr. Naturforsch. Ges. Zürich **85**, 29–34 (1940). [MF 2891]

Kienast, Alfred. Die Green'sche Funktion der Differentialgleichung der Wärmeleitung auf der Kugelfläche. Vierteljschr. Naturforsch. Ges. Zürich **85**, 133–137 (1940). [MF 2890]

The second note contains a direct discussion of the Green

function

$$H(P, Q; t) = (4\pi)^{-1} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) e^{-n(n+1)t}$$

of the equation of the heat conduction on the unit sphere; here P and Q are two points on the unit sphere K, γ their spherical distance, P_n Legendre's polynomial, $t>0$. It is shown that (a) $\Delta H = \partial H/\partial t$ for $t>0$; (b) H is analytic, if P and Q are, on K, $t>0$; (c) $H \to 0$ as $t \to +0$, $P \neq Q$; (d) $\int_k H(P, Q, t) df_P \to 1$ as $t \to +0$. The first note contains the corresponding direct discussion for the linear heat conduction under various boundary conditions, in particular in connection with certain positivity theorems due to Fejér.

G. Szegö (Stanford University, Calif.).

Jaeger, J. C. The solution of boundary value problems by a double Laplace transformation. Bull. Amer. Math. Soc. 46, 687–693 (1940). [MF 2660]

Systematic use of the iterated Laplace transformation is demonstrated here in solving a few variable state problems in heat conduction. The transformations are made with respect to the time variable and one space coordinate. A paper by Carslaw and Jaeger [Proc. Cambridge Philos. Soc. 35, 394–404 (1939); these Rev. 1, 77] gives a method of verifying the solutions found here. The first problem solved is that of the temperatures in the region $x>0$, initially at temperature zero, with the boundary $x=0$ kept at unit temperature. The other problems pertain to the region $x>0$, $0<y<b$. In the first case the region is initially at temperature zero, the boundaries $y=0$ and $x=0$ are kept at zero, and $y=b$ is kept at unit temperature. In the other cases the entire boundary is kept at zero and the initial temperature is taken first as unity, then as x.

R. V. Churchill (Ann Arbor, Mich.).

Carslaw, H. S. and Jaeger, J. C. Some two-dimensional problems in conduction of heat with circular symmetry. Proc. London Math. Soc. (2) 46, 361–388 (1940). [MF 2756]

The formula for the temperature $v(r, t)$ is derived for the solid cylinder $r \leqq a$ of infinite length under each of the following sets of conditions: (1) when $v(r, 0) = v_0$, where v_0 is a constant, and there is "radiation" or insulation at the surface; that is, $\partial v/\partial r + hv = 0$ when $r=a$, where h is a constant and $h \geqq 0$; (2) when the temperature is zero initially except for an instantaneous source of heat on the cylindrical surface $r=r'$, and $v(a, t)=0$; (3) when the surface condition in the problem (2) is replaced by the "radiation" or insulation condition; (4) when the source in problems (2) and (3) is replaced by an instantaneous line source along the axis $r=0$. The temperature formulas are also found for the infinite solid $r \geqq a$ under the sets of conditions (1), (2) and (3) ($h \leqq 0$ here, of course). The problems for the hollow cylinder $a \leqq r \leqq b$ involving essentially the corresponding sets of conditions are also solved. In each case the Laplace transformation with respect to t is used to obtain the solution. To completely establish the result as a solution of the boundary value problem a contour integral form of the result, obtained from the usual inversion integral, is used. This contour integral is identified with the infinite series or real infinite integral obtained as the final form of the temperature function. The authors illustrate this method of verification at the beginning of the paper by solving two familiar and related problems, one for the cylinder $r \leqq a$ with $v(r, 0)=0$ and $v(a, t)=v_0$, and the other for the solid $r \geqq a$ under the same conditions.

R. V. Churchill.

Cinquini-Cibrario, Maria. Sull'analiticità degli integrali di alcune equazioni del primo tipo misto. Ann. Mat. Pura Appl. (4) 19, 51–79 (1940). [MF 3045]

The elliptic-parabolic equation

$$(1) \qquad k^2 y^{2k-2} z_{xx} + z_{yy} = 0, \qquad k=2, 3, \cdots,$$

has associated with it the region D defined by $(x-x_0)^2 + y^{2k} < a^2$. Let D_1 and D_2 be the regions in D for which $y>0$ and $y<0$, respectively. The author solves the Dirichlet problem with continuous boundary values for each of the regions D, D_1 and D_2; each solution is explicitly expressed in terms of a series of particular solutions of (1). Gellerstedt [Ark. Mat. Astr. Fys. 25, no. 10, 1–12 (1935)] solved these problems under more restrictive boundary conditions by means of Green's functions. The main result of the present paper is: a necessary and sufficient condition that an integral of (1) can be continued from D_1 to D_2 (or vice versa) is that it reduces to an analytic function of x on $y=0$. An integral of the hyperbolic-elliptic equation $y^{2k-1} z_{xx} + z_{yy} = 0$ need not be analytic in a region for which $y<0$. The author shows, however, that if an integral is analytic on a segment of the x-axis its extension into the region $y<0$ is an analytic function. Necessary and sufficient conditions for the solution of the Cauchy problem with the x-axis carrying the data is also given for both of the above differential equations.

F. G. Dressel (Durham, N. C.).

Rellich, Franz. Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral. Math. Z. 46, 635–636 (1940). [MF 2787]

Let $u = u(x_1, \cdots, x_n)$ in the domain G be the solution of the differential equation $\Delta u + \lambda u = 0$ satisfying the condition $u=0$ on the boundary Γ of G and $\int\int\int_G u^2 d\tau = 1$. Then λ possesses the representation

$$\lambda = \frac{1}{4} \int\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 \frac{\partial (r^2)}{\partial \nu} d\Omega,$$

where $\partial/\partial\nu$ indicates differentiation with respect to the exterior normal of Γ, and $r^2 = x_1^2 + \cdots + x_n^2$.

K. Friedrichs (New York, N. Y.).

Magnus, Wilhelm. Über eine Randwertaufgabe der Wellengleichung für den parabolischen Zylinder. Jber. Deutsch. Math. Verein. 50, 140–161 (1940). [MF 3074]

The author considers the problem of finding a solution u of the equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

in the interior of a parabola which takes on given boundary values and which behaves at infinity in a certain way. Paul S. Epstein [Über die Beugung au einem ebenen Schirm unter Berücksichtigung des Materialeinflusses, Dissertation München, 1914] has obtained solutions of the form $u = \sum_{n=0}^{\infty} c_n u_n$, where the u_n are of the form $A_n(\xi) \cdot B_n(\eta)$, ξ and η being related to x and y by

$$x = k^{-1}\xi\eta, \quad y = k^{-1}\frac{\xi^2 - \eta^2}{2}.$$

The present author shows that, if the boundary values are analytic and possess certain asymptotic developments at infinity, then there is a unique solution u which behaves properly at infinity and assumes the given boundary values. This solution is exhibited explicitly in the form of a contour integral with respect to the variable n, now taken to be

complex. This form of the solution allows a more careful study of the behavior of the solution at infinity. This problem arises as follows: Given an infinite parabolic cylinder which is a perfect conductor and a system of discreet or continuously distributed set of axes of radiating electrical cylindrical waves, the axes being parallel to the generators of the cylinder. The problem is to determine the electric field function v for the corresponding system of reflected (from the cylinder) waves under the assumption that the field vector of the outgoing system is parallel to the generators and that v is of the form $e^{i\omega t}u$. Then u satisfies (1) and behaves at infinity as is desired provided that the energy flow satisfies a certain limiting condition there.

C. B. Morrey, Jr. (Berkeley, Calif.).

Kupradze, V. und Awazaschwili, D. Eindeutigkeitssatz in der Theorie der Fortpflanzung elektromagnetischer harmonischer Schwingungen im inhomogenen dreidimensionalen Raum. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 35–41 (1940). (Russian with complete German translation) [MF 2017]

The authors give here a proof of the theorem indicated in the title. They state that the generalization to three variables of a proof given by Freudenthal [Compositio Math. 6, 221–227 (1938)] for two variables meets fundamental difficulties. The problem is to prove the nonexistence of a function $u=u(x, y, z)$, $u\not\equiv 0$, satisfying $\Delta u+k_i u=0$ in the domain $\mathfrak{T}_i^3$ and $\Delta u+k_a u=0$ in the complementary part $\mathfrak{T}_a^3$ of the space. At the surface of separation of $\mathfrak{T}_i^3$ and $\mathfrak{T}_a^3$ the solution u satisfies Maxwell's conditions and

$$\lim_{z\to\infty} z\left[\frac{\partial u}{\partial z}-ik_a u\right]=0$$

at infinity; k_i is a complex and k_a a real variable. The authors proved in previous papers [Kupradze Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Math. Inst.] 2, 143–162 (1937); Rec. Math. [Mat. Sbornik] 41, 561–581 (1935); Compositio Math. 6, 228–234 (1938)] that u satisfies a certain integral as well as a functional equation. Using these results they show that the existence of a solution u, $u\not\equiv 0$, implies that the equation $\Delta w+\lambda^2 w=0$, $w\epsilon\mathfrak{T}_i^3$, w const. or $\partial w/\partial n$ const. on the boundary, has complex fundamental values. This is impossible by the classical results.

S. Bergmann (Cambridge, Mass.).

Bateman, H. The solution of harmonic equations by means of definite integrals. Bull. Amer. Math. Soc. 46, 538–542 (1940). [MF 2427]

The author determines functions a, h, b, f of θ, for which for general $F(w)=F(ax^2+2hxy+by^2)$ the expression

$$z=\int_0^\pi F(w)f(\theta)d\theta$$

is a solution of the harmonic differential equation

$$L(z)=z_{xx}+z_{yy}+uz_x\cdot x^{-1}+vz_y\cdot y^{-1}=0; \tag{1}$$

here $u=2k+1$ and $v=2m+1$ are constants. This is achieved by determining a, h, b, f so that the integrand of $L(z)$ becomes an exact differential of the form $d[F'(w)G(x, y, \theta)]$. An example of such an expression is

$$z=\int_0^\pi F(w)\tan^u \tfrac{1}{2}\theta \sin^{v-1} \tfrac{1}{2}\theta d\theta, \tag{2}$$

where $a=\tan^2 \frac{1}{2}\theta$, $b=\sin^2 \frac{1}{2}\theta$, $h=0$, $\Re(u+v)>0$. Writing (2) in a more symmetrical form, we obtain solutions of (1) of the form

$$z=\int_C F(w)(\alpha^2+v)^{-k-1}(\beta^2+v)^{-m-1}dv,$$

where $w=x^2(\alpha^2+v)^{-1}+y^2(\beta^2+v)^{-1}$, and where the contour C starts and ends at infinity and encloses one of the points $v=-\alpha^2$ or $v=-\beta^2$.

F. John (Lexington, Ky.).

Korytnikova, N. The influence of the water basins on the thermal conditions of the neighboring parts of the earth crust. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 1940, 17–32 (1940). (Russian. English summary) [MF 2677]

The author investigates a stationary distribution of temperature around a water basin of ellipsoidal shape. In mathematical terms he is concerned with the solution $u(x, y, z)$ of $\Delta u=0$ which equals a given constant T_0 on the portion $z<0$ of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and on the portion of $z=0$ outside the ellipsoid; moreover it is assumed that $u\sim T_0+\gamma z$ for large values of z, where γ is a constant. Using elliptic coordinates, the solution can be represented by a definite integral. The author studies numerically the flow of temperature on the bottom of the basin.

W. Feller (Providence, R. I.).

Keldych, M. et Lavrentieff, M. Sur une évaluation pour la fonction de Green. C. R. (Doklady) Acad. Sci. URSS (N.S.) 24, 102–103 (1939). [MF 2913]

Let D be a finite domain in 3-dimensional space, bounded by a closed surface S, with Green's function $G(P, Q)$. Rosenblatt [C. R. Acad. Sci. Paris 201, 22–24 (1935)] has shown that, when S has a continuous curvature, $G(P, Q)$ satisfies the inequality $G(P, Q)<A\sigma s/\overline{PQ}^3$, where s and σ are the respective distances of P and Q from S and A is a constant depending on D. In the present paper this result is established for the case that S is an arbitrary "surface of Liapounoff," a surface with tangent plane at every point and such that, for every two points M_1 and M_2 on S, the (appropriately defined) angle ψ between the normals at M_1 and M_2 satisfies the inequality $\psi<K(\overline{M_1M_2})r$, where K and r depend only on S and $0<r<1$. The authors' definition of a surface of Liapounoff is not precise with respect to the definition of the angle ψ, and the paper contains the apparently inadvertent statement that Rosenblatt [loc. cit.] has established the inequality in question for a surface S with bounded curvature.

J. W. Calkin (Chicago, Ill.).

Privaloff, I. I. et Kouznetzoff, P. Sur les problèmes limites et les classes différentes de fonctions harmoniques et subharmoniques définies dans un domaine arbitraire. Rec. Math. [Mat. Sbornik] N.S. 6 (48), 345–376 (1939). (Russian. French summary) [MF 1944]

In this paper theorems of Fatou's type for subharmonic functions of two and three variables are proved. In the case of two variables the authors consider four classes of functions U subharmonic in a domain D bounded by a rectifiable curve Γ. These are functions: (a) for which

$$\textstyle\int_{\Gamma_\lambda}(U^+)^q\partial G/\partial n\, ds<K, \qquad \lambda>0,$$

(b) for which

$$\textstyle\lim_{\lambda\to 0}\int_{\Gamma_\lambda}U^+\partial G/\partial n\, ds=\int_\Gamma U^+\partial G/\partial n\, ds,$$

(c) for which

$$\lim_{\lambda\to 0} \int_{\Gamma_\lambda} U\, \partial G/\partial n\, ds = \int_\Gamma U\, \partial G/\partial n\, ds$$

and, finally, (d) those with

$$\int_{\Gamma_\lambda} U^n \partial G/\partial n\, ds < K, \qquad \lambda > 0,$$

for which $\lg U$ is also subharmonic. Γ_λ means the line $G(x, 0)=\lambda$ and G Green's function of D. Using the mapping theorem and the previous results of Privaloff [Rec. Math. [Mat. Sbornik] N.S. 3 (45), 3–25 (1938); Bull. Acad. Sci. URSS [Izvestia Akad. Nauk. SSSR] (N.S.) 18, 5–7 and 507–510 (1938)], the authors give necessary and sufficient conditions under which U belongs to one of the indicated classes; they derive the representation

$$U(P) = -\int\int_D G(P, Q) d\mu(Q) + (2\pi)^{-1} \int_\Gamma U(Q) \partial G(P, Q)/\partial r_Q\, ds$$

for the functions indicated under (c) and analogous integral formulas for other cases, from which it follows that U possesses boundary values almost everywhere.

In the case of functions of three variables, the boundary S of D is supposed to satisfy Liapounoff's conditions. The authors prove that a function H harmonic in D and representable by the formula of Green-Lebesgue or Green-Stieltjes possesses boundary values almost everywhere, and they give conditions under which a harmonic function can be represented by these formulas. Furthermore, they consider subharmonic functions $U(P)$, $(P)=(x, y, z)$, especially certain classes of them, analogous to those considered in the case of two variables. They give conditions under which U can be represented in the form

$$U(P) = -\int\int\int_D G(P, Q) d\mu(Q) + (4\pi)^{-1} \int\int_S \partial g(P, Q)/\partial n_Q d\psi(e)$$

and has boundary values almost everywhere. μ and ψ are the corresponding mass distributions. The proofs are based on an investigation of the behavior of Green's function in the neighborhood of a point of the boundary S and on some previous results of Privaloff. It may be worthwhile to indicate that there is a connection between the above results on harmonic functions and a paper of de la Vallée Poussin [Ann. Scuola Norm. Super. Pisa (2) 2, 167–197 (1933)].

S. Bergmann (Cambridge, Mass.).

Monna, A. F. Extension du problème de Dirichlet pour ensembles quelconques. Nederl. Akad. Wetensch., Proc. 43, 497–511 (1940). [MF 2257]

This paper continues the work of Brelot [Acad. Roy. Belgique. Bull. Cl. Sci. 25, 125–137 (1939); these Rev. 1, 238] and of the author [Nederl. Akad. Wetensch., Proc. 42, 745–752 (1939); these Rev. 1, 122] on the construction of functions solving the Dirichlet problem, extending the methods to general sets E. For a continuous subharmonic function ϕ, defined on all space, the interior solution $\underline{u}(P)$ (exterior solution $\bar{u}(P)$) is the superior (inferior) envelope of solutions for ϕ and all closed sets contained in (open sets containing) E. For a continuous ϕ which is the difference of two subharmonic functions, $\underline{u}=\underline{u}'-\underline{u}''$, $\bar{u}=\bar{u}'-\bar{u}''$, and these solutions are thus uniquely determined; they are linear functionals of ϕ and may be extended to the case where ϕ is an arbitrary continuous function. In these terms the author describes various interior and exterior properties. Thus a frontier point Q of E is interiorly stable if $\underline{u}(Q)=\phi(Q)$, interiorly regular if $\lim_{P\to Q} \underline{u}(P)=\phi(Q)$, for every continuous ϕ. The connection of these ideas with the criteria for a set being sharp-effilé, drawn-out, at a point [Brelot, C. R. Acad. Sci. Paris 209, 828–830 (1939); these Rev. 1, 121] is immediate. Being linear functionals, the $\bar{u}$ and $\underline{u}$ can be written as Stieltjes integral averages of ϕ on the frontier of E, with respect to mass distributions $\bar{\mu}$ and $\underline{\mu}$, which are arrived at by generalizations of the sweeping-out process, that is, exterior and interior extremalization, of unit mass. For measurable sets E the $\bar{\mu}(\underline{\mu})$ has no mass on the sets of points exteriorly (interiorly) unstable or irregular.

G. C. Evans (Berkeley, Calif.).

Evans, G. C. Surfaces of minimal capacity. Proc. Nat. Acad. Sci. U. S. A. 26, 489–491 (1940). [MF 2634]

If s is a closed curve in space and S is a surface of which s is the complete boundary, it is asked if among all S there exists one of minimum capacity. The author considers an s which is of zero capacity and such that a large sphere containing s in its interior can be transformed into itself by a one-to-one transformation ξ in such a manner that s is transformed into a circle σ and the surface points of the sphere remain invariant. In the course of the proofs use is made of this condition on s by the introduction of "tori" about s; that is, surfaces obtained by applying ξ on tori about σ. These surfaces have necessarily the topological properties of the torus. If S is a solution of the problem, then at all points of a sufficiently smooth part of S, the following relation must hold:

$$\frac{dV(Q)}{dn+} = \frac{dV(Q)}{dn-}, \tag{1}$$

where $V(Q)$ is the conductor potential of S at Q, and $n+$ and $n-$ denote the two oppositely directed normals to S at Q. The principle theorem of the paper states the existence of a surface S satisfying (1) at all points of its sufficiently smooth parts, and analytic except for nodal lines and points. In the proof, a torus about s is selected and a two valued harmonic function set up which is equal to 1 on the torus and equal to 2 or 0 at ∞, depending on the branch taken. By allowing the torus to shrink to s, a function $v(P)$ is obtained, and the level surface $S: v(P)=1$ is shown to satisfy (1) and to have s for its complete boundary. The conductor potential of S is the minorant of the two values of $v(P)$.

J. W. Green (Rochester, N. Y.).

Evans, G. C. Surfaces of minimum capacity. Proc. Nat. Acad. Sci. U. S. A. 26, 664–667 (1940). [MF 3147]

In the paper reviewed above the author demonstrated the existence of a unique piecewise smooth and regular surface S spanning a given space curve s, and such that if $V(Q)$ is the conductor potential for S, and $n+$ and $n-$ the two normals to S at a point, then $dV/dn+ = dV/dn-$, this being the necessary condition that S have the least capacity among all such surfaces spanning s. Precise conditions on s and S were given in the paper cited. In the present paper, it is shown that the surface S thus obtained actually realizes the least capacity among all surfaces S' which form caps for S; that is, surfaces S' such that (1) every point of s is a limit of points of S', (2) S' is bounded and composed of finitely many smooth and regular pieces, except in the neighborhood of s, and (3) every closed curve looping s contains a point of S'. In the proof it is first assumed that s and S, as well as S', the surface with which S is being compared, are sufficiently smooth that Green's theorem can be

applied. It is then shown by expressing capacity in terms of the Dirichlet integral that the capacity of S' is greater than that of S unless $S \equiv S'$. If the conditions of smoothness are not satisfied, then s is surrounded by a smooth tube, or "torus," $\sum_n$. By considering the smooth surfaces thus obtained, and allowing $\sum_n$ to tend to s, the author is able to prove the same inequality for the capacities of S and S' as hold in the smooth case. *J. W. Green.*

Sen, Bibhutibhusan. Note on the bending of thin uniformly loaded plates bounded by cardioids, lemniscates and certain other quartic curves. Z. Angew. Math. Mech. **20**, 99–103 (1940). [MF 2591]

The differential equation $\Delta\Delta u = \text{const.}$ with the boundary condition (clamped edge) $u = \partial u/\partial \nu = 0$ (ν being the normal) is solved explicitly when the boundaries are cardioids, lemniscates, inverses of ellipses and elliptic limacons.

W. Feller (Providence, R. I.).

Calculus of Variations

Smiley, M. F. The Jacobi condition for extremaloids. Duke Math. J. **6**, 425–427 (1940). [MF 2326]

The author gives a definition of conjugate point for extremaloids having a finite number of corners in terms of a very general type of "accessory pseudo-extremaloid." This definition, like a related one of McShane applicable to extremals, permits simultaneous consideration of the parametric and non-parametric problems. The author shows that his definition is equivalent to that of the reviewer for parametric problems and reduces to that of Reid for non-parametric problems. *L. M. Graves* (Chicago, Ill.).

Mancill, J. D. The Jacobi condition for unilateral variations. Duke Math. J. **6**, 341–344 (1940). [MF 2318]

The author gives a proof of the Jacobi condition for parametric problems of the calculus of variations in the plane which is applicable to the case when the minimizing extremal lies partly or entirely along the boundary of the region where admissible curves must lie. The proof is geometric in character, but eliminates the usual restrictive assumption that the envelope has a regressive branch at its point of contact with the minimizing curve.

L. M. Graves (Chicago, Ill.).

McShane, E. J. Generalized curves. Duke Math. J. **6**, 513–536 (1940). [MF 2713]

The present paper is the first of three papers whose principal object is to establish existence theorems for Bolza problems in the calculus of variations. In this first paper the author outlines the methods to be used in the three papers and studies in detail the properties of generalized curves which will be needed later. Generalized curves were invented by L. C. Young. One of the most useful properties of these curves is that, by a suitable choice of a metric, the integrals of the calculus of variations can be made to be continuous functions of these curves without sacrificing compactness of closed sets of curves of bounded lengths. In the closing sections of the paper the author establishes existence theorems for the problem of Bolza in the class of generalized curves and shows that under certain conditions the minimizing generalized curve is an ordinary curve. This includes the known existence theorems of this type. More general results will be given in later papers.

M. R. Hestenes (Chicago, Ill.).

McShane, E. J. A remark concerning sufficiency theorems for the problem of Bolza. Bull. Amer. Math. Soc. **46**, 698–701 (1940). [MF 2662]

The usual sufficiency theorems for the problem of Bolza assume that the curve in question satisfies the multiplier rule, the transversality condition and the strengthened conditions of Weierstrass, Clebsch and Jacobi for a set of multipliers with $\lambda_0 > 0$ (in which case λ_0 may be taken as 1). The author remarks that the sufficiency theorems also hold if the assumption is merely $\lambda_0 \geqq 0$. This is an immediate consequence of the observation that, if the curve in question satisfies all the strengthened conditions for a set of multipliers with $\lambda_0 = 0$, then no neighboring curve can satisfy the side conditions of the original problem. Just as trivially, it is shown, in the case $\lambda_0 = 0$, that the extremal is a normal minimizing curve for a certain Mayer problem naturally associated with the original problem. *M. Shiffman.*

Hölder, Ernst. Reihenentwicklungen aus der Theorie der zweiten Variation. Abh. Math. Sem. Hansischen Univ. **13**, 273–283 (1940). [MF 2385]

Referring to his paper in Acta Math. **70**, 193–242 (1939) for fuller details, the author discusses the second variation of a general calculus of variations problem, taken in the form $\int(-H+\dot{q}_i p_i)dt$, by considering an associated boundary value problem. The procedure is, in its general outline, the same as that followed in recent years by Bliss, Morse, Reid and others. The point in which this treatment differs from the work of these authors is mainly the introduction of a "Green tensor," leading to a transformation of the second variation to a form which yields almost immediately the conclusion that for the non-negative character of the second variation it is necessary and sufficient that the least positive characteristic number $\mu^{(1)}$ of the boundary value problem shall be not less than 1. It follows furthermore that a minimum is obtained if $\mu^{(1)} > 1$, that a minimum can not take place when $\mu^{(1)} < 1$, while the case $\mu^{(1)} = 1$ remains undecided. *A. Dresden* (Swarthmore, Pa.).

Bardell, Ross H. The inequalities of Morse when the maximum type is at most three. Bull. Amer. Math. Soc. **46**, 242–245 (1940). [MF 1825]

This paper studies a simple regular integral of the calculus of variations in parametric form. Extremals are sought joining two fixed points P and Q in an extremal convex region R. The author wishes to establish the Morse relations between the type numbers of extremals joining P to Q without using the general theory of critical points. His methods involve a study of the way the intersections of two extremals through P vary with their initial directions, and as developed depend upon his assumption that the region is plane and the problem analytic. Other restrictions are as follows. No point of the boundary B is a point envelope of the family H of extremals through P; on each extremal of H there is a first point of intersection with B; no extremal of H intersects itself between P and B; the maximum type of each extremal is at most three. The method yields interesting properties of "pairs" of extremals joining P to Q.

M. Morse (Princeton, N. J.).

Morrey, Charles B., Jr. Existence and differentiability theorems for the solutions of variational problems for multiple integrals. Bull. Amer. Math. Soc. **46**, 439–458 (1940). [MF 2410]

The present paper is an account of a Symposium Lecture given before the American Mathematical Society. It contains principally a summary of recent researches by the author. The object of these researches is "first, to demonstrate, by direct methods, the existence of solutions, perhaps in some generalized sense, for a wide class of variational problems for multiple integrals, and second, to investigate further differentiability properties of the generalized solutions thus obtained." In order to obtain these results the author uses a larger class of admissible functions than those used heretofore. The functions used possess generalized derivatives and were introduced by G. C. Evans. Existence theorems are established for an N-tuple integral of the form $\int_G f(x, z, p)dx$, where x is an N-tuple, z is P-tuple and p is an NP-tuple denoting the generalized derivatives of the functions $z(x)$ with respect to the x's. The author assumes that the function $f(x, z, p)$ is continuous, is convex in p for each set (x, z) and is such that an inequality of the form $f(x, z, p)\geqq\phi(p)$ holds, where $\varphi(p)$ is a convex function for which $\lim_{|p|\to\infty}|p|^{-1}\varphi(p)=+\infty$, the symbol $|p|$ denoting the norm of p. Restricting himself to the case when $N=2$ and making further hypotheses on the integrand, the author obtains continuity and differentiability theorems for minimizing functions $z(x)$. These results are of great interest and are best summarized in the paper here reviewed. An extensive study of the properties of the admissible functions used is given. *M. R. Hestenes* (Chicago, Ill.).

Kneser, Hellmuth. Homogene Funktionen auf der Grassmannschen Mannigfaltigkeit. Bull. Soc. Math. Grèce **20**, 101–103 (1940). [MF 2633]

Sakellariou [Monatsh. Math. Phys. **48**, 314–321 (1939); these Rev. **1**, 78] proved some interesting results concerning the external surfaces of a regular variational problem of the form

$$(1)\quad \delta\int\int F(p^{ik})dudv=0,\quad p^{ik}=\frac{\partial(x^i, x^k)}{\partial(u, v)},\quad p^{ik}=-p^{ki},$$

in which F is of class C'', is positively homogeneous of the first degree in the p^{ik}, and satisfies the equations

$$(2)\qquad \frac{\partial F}{\partial p^{ik}}\frac{\partial F}{\partial p^{lm}}+\frac{\partial F}{\partial p^{il}}\frac{\partial F}{dp^{mk}}+\frac{\partial F}{\partial p^{im}}\frac{\partial F}{\partial p^{kl}}=0,\qquad i, k, l, m=1, \cdots, n.$$

It is easy to see that the integral in (1) is unchanged if F is replaced by another function Φ which coincides with F on the so-called Grassmann manifold which is defined by the relations

$$p^{ik}+p^{ki}=0,\ p^{ik}p^{lm}+p^{il}p^{mk}+p^{im}p^{kl}=0,\ i, k, l, m=1, \cdots, n.$$

In the present paper the author demonstrates the following result: Let F be any function of the above type which is never zero (unless all the $p^{ik}=0$) but which does not necessarily satisfy equations (2). Then there exists a function Φ coinciding with F on the Grassmann manifold and there exist quantities u_{ik} determined by Φ such that the derivatives $\partial\Phi/\partial p^{ik}$ may be replaced by the quantities u_{ik} in the equations of variation and such that the equations:

$$F=\sum u_{ik}p^{ik},\quad dF=\sum u_{ik}dp^{ik},$$
$$u_{ik}+u_{ki}=0,\quad u_{ik}u_{lm}+u_{il}u_{mk}+u_{im}u_{kl}=0$$

all hold on the Grassmann manifold. It is to be noted that the last equation is the analogue of (2) above.

C. B. Morrey (Berkeley, Calif.).

Patterson, William A. Inverse problems of the calculus of variations for multiple integrals. Bull. Amer. Math. Soc. **46**, 502–511 (1940). [MF 2419]

This paper asks under what conditions the integral hypersurfaces $z=z(x^a)$ of a partial differential equation

$$F\equiv A^{a\beta}(x^a|z|p_a)p_{a\beta}+B(x^a|z|p_a)=0$$

will be the extremal hypersurfaces of a variation problem

$$I\equiv\int_{(n)} f(x^a|z|p_a)dx^1\cdots dx^n=\min.$$

It is well known that $F=0$ will be the Euler-Lagrange partial differential equation for some I if, and only if, $F=0$ has a self-adjoint equation of variation. This paper finds that there is at most one multiplier M such that $M\cdot F=0$ has a self-adjoint equation of variation. If the $A^{a\beta}$ are constants and B does not contain p_a in general the equation $F=0$ has the multiplier $M\equiv1$. The paper then discusses in detail the equation $A^{a\beta}(p_1, p_2)p_{a\beta}=0$, $\alpha, \beta=1,2$, for which there does not always exist a multiplier. *E. W. Titt.*

Morse, Marston. The first variation in minimal surface theory. Duke Math. J. **6**, 263–289 (1940). [MF 2314]

Let $\gamma_0, \gamma_1, \cdots, \gamma_n$ be a set of simple closed curves in Euclidean m-space such that no two of the curves intersect. Let B be a connected region of the u, v-plane bounded by $n+1$ circles $C_0, C_1, \cdots, C_n$, and let θ_k be an angular parameter on C_k. The author considers surfaces S satisfying the following conditions: S is a continuous image of B and its boundary; S is defined by equations $x_i=f_i(u, v)$, where the functions $f(u, v)$ are harmonic in the interior of B; the curves C are mapped in a continuous monotone fashion on the corresponding curves γ. Such surfaces are called admissible. Admissible surfaces may be varied by changing the positions and the radii of the circles C or by changing the mapping of the circles C on the curves γ. The author considers the Douglas-Dirichlet integral D defined by S, and proves that if D is finite a necessary and sufficient condition that S be a minimal surface is that the first variation of this integral in the admissible family defined above is zero. This theorem is an extension of the one which led Douglas to introduce the functional D, and it is fundamental in minimal surface theory. It has been proved previously only in case S was known to be a minimizing extremal surface of D.

The proof briefly is as follows. Classical methods establish it if the mapping of the C's on the γ's is sufficiently smooth. Next the author shows that the functional D is continuous in all possible arguments, and the proof follows by a simple limiting process. The continuity of D in the case of a single contour is proved by straightforward computation, and in the case of many contours by noticing that the difference between D and the sum of the Douglas-Dirichlet functional for the $n+1$ surfaces of the type of a circular disc obtained by considering the mappings of the C's on the γ's singly is well behaved. *C. B. Tompkins* (Princeton, N. J.).

Douglas, Jesse. The higher topological form of Plateau's problem. Ann. Scuola Norm. Super. Pisa (2) **8**, 195–218 (1939). [MF 2037]

In previous papers the author has stated the problem of Plateau in its topologically general form and has presented two methods of solution of the problem. These methods

utilized, respectively, the Green's function of a general Riemann surface and the theory of multiple theta functions. Here the outstanding features of the two methods are set forth and compared; details of proof are omitted in order that the essentials may be more clearly seen.

E. J. McShane (Charlottesville, Va.).

Courant, R. The existence of minimal surfaces of given topological structure under prescribed boundary conditions. Acta Math. **72**, 51–98 (1940). [MF 2015]

The author presents a solution of the problem of Plateau (the determination of a minimal surface bounded by a given Jordan curve), of the problem of Douglas (the determination of a minimal surface of a prescribed topological structure bounded by k prescribed Jordan curves), and of a more general problem, which we might call the problem of Courant, in which part or all of the boundary is free on prescribed continuous manifolds. It is assumed throughout that the given boundary is capable of spanning surfaces of the prescribed structure and with finite area. The details of the solution of the problem of Courant are given only for the typical case of a doubly-connected minimal surface one of whose boundaries is free on a given closed surface M, while the other is a given Jordan curve Γ monotonically described. It is shown that the variational problem of minimizing the Dirichlet vector integral, which actually is equivalent to minimizing the area, has a solution provided M and Γ together bound a doubly-connected surface of area less than that of the simply-connected surface of least area bounded by Γ. It is then shown that the solution, which first of all must be given by a harmonic vector, is a minimal surface; a simple proof based on conformal mapping and also a proof independent of conformal mapping are given. The minimal surface is shown to be orthogonal, in a weak sense, to M. The methods are largely simplifications of those previously used by the author. *E. F. Beckenbach.*

NUMERICAL AND GRAPHICAL METHODS

Sebastião e Silva, José. On the numerical resolution of algebraic equations. Portugaliae Math. **1**, 303–332 (1940). (Portuguese) [MF 2892]

Collatz, L. Das Hornersche Schema bei komplexen Wurzeln algebraischer Gleichungen. Z. Angew. Math. Mech. **20**, 235–236 (1940). [MF 3114]

Horner's method can be used to approximate the complex roots of an algebraic equation in exactly the same way as for real roots, but the computation is four times as great due to the four real products involved in each product of complex numbers. The author's modification reduces this work by half and avoids the use of complex numbers almost entirely, provided the given coefficients are real. This is accomplished by reducing the given equation by means of the quadratic equation with real coefficients which is satisfied by the trial root and its conjugate. The method yields exactly the same correction to be applied to the trial root as does Horner's method. *P. W. Ketchum* (Urbana, Ill.).

San Juan, R. Complements to Gräffe's method for the solution of algebraic equations. Revista Mat. Hisp.-Amer. (3) **1**, 1–14 (1939). (Spanish) [MF 1668]

In the theory of Graeffe's method it is important to be able to ascertain if the n roots x_ν of a given equation (1) $a_0x^n+\cdots+a_n=0$ satisfy the inequalities (2) $|x_1|\geqq|x_2|\geqq\cdots\geqq|x_m|>|x_{m+1}|\geqq\cdots\geqq|x_n|$. Let $M=\max_r |a_{m+r}/a_m|^{1/r}$, $r=0, 1, \cdots, n-m$, $N=\min_s |a_m/a_{m-s}|^{1/s}$, $s=1, \cdots, m$. It is shown that the inequality (3) $10M<N$ implies (2). While (3) is sufficient but not necessary for (2) to hold, it is shown that (2) implies the inequality analogous to (3) for the coefficients of a Graeffe transform of (1) of sufficiently high order. There follows a discussion of the relative errors committed in computing the roots of (1) by splitting the Graeffe transform in the usual way in two equations of degree m and $n-m$, respectively. *I. J. Schoenberg.*

Reiersøl, Olav. A method for recurrent computation of all the principal minors of a determinant, and its application in confluence analysis. Ann. Math. Statistics **11**, 193–198 (1940). [MF 2340]

The pivotal method is adapted to the calculation of all the principal minors of a given order in a determinant by recurrence formulae from the values of the principal minors of lower orders. A numerical illustration is given and it is pointed out that this method gives the coefficients of the characteristic polynomial of a matrix for degree less than 10 in fewer operations than does a method due to Horst.

C. C. Craig (Ann Arbor, Mich.).

Greenspan, Martin. Approximation to a function of one variable from a set of its mean values. J. Research Nat. Bur. Standards **23**, 309–317 (1939). [MF 2147]

Methods of measurement to determine the values of a function of a single variable may be such that only mean values of the function over intervals of the argument are obtained. For approximating to the values of the function from such data, two principal formulae are derived: a central-difference formula and a descending-difference formula. Some examples of their application are given. The justification for the use of these formulae rests on the fact that the distributions encountered in practice can usually be represented sufficiently well by polynomials of low degree.

R. M. Foster (New York, N. Y.).

Bleick, W. E. A least squares accumulation theorem. Ann. Math. Statistics **11**, 225–226 (1940). [MF 2346]

The following theorem is proved: If $A^*(x)$ and $B^*(x)$ are polynomials of the same degree which are least squares representations of the functions $A(x)$ and $B(x)$, respectively, for the values $x_1, x_2, \cdots, x_p$, then $\sum A^*(x_i)B(x_i)=\sum A(x_i)B^*(x_i)=\sum A^*(x_i)B^*(x_i)$. *W. Feller.*

Kimball, Bradford F. Orthogonal polynomials applied to least square fitting of weighted observations. Ann. Math. Statistics **11**, 348–352 (1940). [MF 2837]

Let x be the independent variable which can take n consecutive integral values, say from 1 to n. Denote by y_x the observed value of the dependent variable y corresponding to the value x of the independent variable. In fitting a polynomial to the observations $y_1, \cdots, y_n$ by the method of least squares, the computational work can significantly be reduced by using orthogonal polynomials. In case that the observations have equal weights the method of fitting by orthogonal polynomials has been thoroughly worked out and simplified by Fisher, Aitken and others. If the observations have unequal weights the method of fitting is somewhat

involved. In this paper the author proposes an improvement in the technique of fitting by orthogonal polynomials in the case of observations with unequal weights. A system of polynomials $\Phi_r(x)$ $(r=0, 1, 2, 3, \cdots)$ of degree r in x are considered (called orthogonal polynomials), which satisfy the condition

$$\sum_{s=1}^{n} w_s \Phi_r(x)\Phi_s(x)=0, \qquad \text{if } r\neq s.$$

Here w_s denotes the weight of the observation y_s. To construct the polynomials, one may write them in the form

$$\Phi_0(x)=f_0(x)=\text{constant}, \quad \Phi_r(x)=f_r(x)-\sum_{i=0}^{r-1} h_i\Phi_i(x),$$

where $f_r(x)$ denotes an arbitrary polynomial of degree r and

$$h_i=\sum_{s=1}^{n} w_s f_r(x)\Phi_i(x)/\sum_{s=1}^{n} w_s[\Phi_i(x)]^2.$$

The author shows that, if we choose

$$f_r(x)=\frac{x(x-1)\cdots(x-r+1)}{r!},$$

the technique of fitting can be simplified. *A. Wald.*

Schumann, T. E. W. A mechanical appliance for the smoothing of time series. Philos. Mag. (7) **30**, 39–48 (1940). (2 plates) [MF 2566]

A row of vertical rods floating in mercury represent by their heights the ordinates under consideration. They are adjusted to represent the given ordinates; then a wire passing through holes at their tops is drawn taut, bringing them closer together. The amount of smoothing depends on the tension of the wire. A test of the device shows accuracy to about two percent. An improved model is under construction. *A. Blake* (Washington, D. C.).

Green, Harold D. Square root extractor. Rev. Sci. Instruments **11**, 262–264 (1940). [MF 2615]

This paper describes a drawing-board device which will replot on a linear scale curves that have been drawn on a squared scale, by the recorders of square-law measuring instruments. *S. H. Caldwell* (Cambridge, Mass.).

Eigermann, Alfred. Détermination analytique des paramètres homographiques dans les nomogrammes à points alignés. Revue Sci. (Rev. Rose Illus.) **78**, 139–145 (1940). [MF 2196]

Expository article.

Christen, H. et Linder, A. Une application de la nomographie au système complet de rentes viagères de Blaschke-Gram. Skand. Aktuarietidskr. **1940**, 15–24 (1940). [MF 2962]

Agostini, Amedeo. Sopra gli abachi ad allineamento. Boll. Un. Mat. Ital. (2) **2**, 360–363 (1940). [MF 2989]

Shor, I. B. Graphical methods in statics and kinematics of complicated space systems. Uspekhi Matem. Nauk **7**, 268–315 (1940). (Russian) [MF 2172]

Wilsing, H. Rechenscheibe zur Umwandlung von alter in neue Winkelteilung und umgekehrt. Allg. Vermessgs.-Nachr. **52**, 258–262 (1940). [MF 3006]

Hazen, H. L. and Brown, G. S. The cinema integraph. A machine for evaluating a parametric product integral. Appendix by W. R. Hedeman, Jr. J. Franklin Inst. **230**, 19–44, 183–205 (1940). [MF 2469]

The cinema integraph is based on optical principles and is designed for the evaluation of integrals of the form $\int_a^b g(x)f(x)dx$. The functions $g(x)$ and $f(x)$ may be in analytic, tabular or graphical form; they are plotted on photographic film for use by the machine. Automatic controls make the machine capable of evaluating the parametric integrals $F(y)=\int_a^b f(x, y)g(x)dx$ and $F(y)=\int_a^b f(y\pm x)g(x)dx$. The former must be evaluated on a point-by-point basis, while a continuous solution may be obtained for the latter. Solutions are produced in graphical and tabular form. The majority of problems can be solved with an accuracy of 1 percent of full scale, while the time required for solution is much less than that required by numerical methods. Detailed description is given of the machine and of the procedure followed in using it. Several typical problems are worked out in numerical form. Through evaluation of these integrals, the cinema integraph is useful in many fields of engineering and science. Applications are found in the Fourier transform, the superposition theorem, the Poisson type of integral equation and in periodogram and correlation analysis. The expansion of a function in terms of orthogonal or bi-orthogonal functions, as the Fourier series and Legendre, Laguerre, Hermitian and Tschebyscheff polynomials is another field of application. Integral equations of various types may be solved by using methods of successive approximation. *S. H. Caldwell* (Cambridge, Mass.).

Hartree, D. R. The Bush differential analyser and its applications. Nature **146**, 319–323 (1940). [MF 2744]

Heinrich, Helmut. Bemerkungen zur graphischen Integration. Z. Angew. Math. Mech. **20**, 121–123 (1940). [MF 2595]

The author discusses the accuracy of two well-known processes of graphical integration of the differential equation $y'=f(x, y)$, the one based on the formula

$$\Delta y=\tfrac{1}{2}hf_0+\tfrac{1}{2}hf(x_0+h,\ y_0+hf_0),$$

the other on the formula

$$\Delta y=hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}hf_0),$$

for the approximate calculation of the increment Δy as x varies from x_0 to x_0+h. He examines in some detail various methods of iteration based on the above formulas and designed to yield an improved approximation.

W. E. Milne (Corvallis, Ore.).

Sadowsky, Michael. A formula for approximate computation of a triple integral. Amer. Math. Monthly **47**, 539–543 (1940). [MF 3079]

The problem is to obtain an approximate formula for computing the integral

$$I=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} u(x, y, z)dxdydz$$

using only values of u at points on the surface of the bounding cube. The author obtains

$$I=\frac{4}{225}[91\sum u_6-40\sum u_{12}+16\sum u_{24}].$$

Here $\sum u_6$ denotes the sum of the u's at the centers of the six faces, $\sum u_{12}$ denotes the sum of the u's at the mid-points

of the 12 edges, and $\sum u_{24}$ denotes the sum of the u's at the 24 points on the diagonals of the faces at distance $\frac{1}{2}\sqrt{5}$ from the center of the face.

It is shown: (a) that no surface type formula can be exact if u is a polynomial of degree 6 or higher; (b) that the formula above is exact for all polynomials of degree 5 or less; (c) that any other formula using 42, or less, surface points is essentially less accurate than the one given.

W. E. Milne (Corvallis, Ore.).

Zurmühl, Rudolf. Zur numerischen Integration gewöhnlicher Differentialgleichungen zweiter und höherer Ordnung. Untersuchungen zu den Verfahren von Blaess und Runge-Kutta-Nyström. Z. Angew. Math. Mech. **20**, 104–116 (1940). [MF 2592]

After a brief explanation of Blaess's procedure for the approximate solution of differential equations of the second order, the author investigates the magnitude of the error, using as a criterion the degree of coincidence with Taylor's series. For the general case $y''=f(t, y, y')$, it is shown that the approximation agrees with Taylor's series up to the term in h^4 (h is the length of the "step"), while for the special equation $y''=f(t, y)$ agreement is secured to the term in h^5. Building upon Blaess's computational set-up the author gives a convenient form for handling Nyström's adaptation of the Runge-Kutta procedure for second order equations. This method requires four substitutions in the equation for each step. He also derives formulas, similar in spirit to those of Runge-Kutta, for the direct treatment of equations of the third order. These take on a simpler form when either y' or y'' is missing. They also require four substitutions in the differential equation for each step. The applications of the methods to actual calculation is illustrated by several examples.

W. E. Milne.

Mikeladze, Sch. Über die Integration von Differentialgleichungen mit Hilfe der Differenzenmethode. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 627–642 (1939). (Russian. German summary) [MF 2100]

A procedure for the numerical calculation of the solution of the equation $y^{(n)}=\varphi(x, y^{(k)})$, $k=1, 2, \cdots, n-1$, $y^{(k)}=dy^k/dx^k$ is described in this paper. The values of $y^{(k)}$, $k=0, 1, 2, \cdots, n$, at p equidistant points ih, $i=0, 1, \cdots, (p-1)$ being given, the author indicates a rule for finding the values of $y^{(k)}$ at the point ph. The author here uses the differences $\Delta^{(s)}y^{(k)}(ih)$. His method is based on the formula

$$y^{(k)}(-\alpha h)=\sum_{\lambda=0}^{n-k}(-\alpha)^\lambda h^\lambda y^{(k+\lambda)}(0)/\lambda!+(-h)^{n-k}\sum_{\lambda=1}^{r}\beta_\lambda\Delta^\lambda y^{(n)}(-\lambda h)+R,$$

where β_λ are constants and R the remainder. By substituting special values in his formula, the author obtains the formulas of Adams, Störmer, Laplace and Falkner. By a slight modification and specialization, he gets formulas of Cowell, Florinsky and Vetshenkin. A procedure for estimating the accuracy of the calculations is given. The present reviewer would like to point out that a method of finding the exact range of errors for this method in the case of equations of the first degree was given by von Mises [Z. Angew. Math. Mech. **10**, 81–92 (1930)] and for still other systems of differential equations by Schulz [Z. Angew. Math. Mech. **12**, 44–59 (1932)].

S. Bergmann (Cambridge, Mass.).

Blaisdell, B. Edwin. The physical properties of fluid interfaces of large radius of curvature. I. Integration of Laplace's equation for the equilibrium meridian of a fluid drop of axial symmetry in a gravitational field. Numerical integration and tables for sessile drops of moderately large size. II. Numerical tables for capillary depressions and meniscus volumes in moderately large tubes. III. Integration of Laplace's equation for the equilibrium meridian of a fluid drop of axial symmetry in a gravitational field. Approximate analytic integration for sessile drops of large size. J. Math. Phys. Mass. Inst. Tech. **19**, 186–245 (1940). [MF 2641]

The differential equation mentioned in the title is

$$\frac{y''}{(1+y'^2)^{3/2}}+\frac{y'}{x(1+y'^2)^{1/2}}=2(h_0+y),$$

where h_0 is a free parameter; x, y are the Cartesian coordinates of the meridian measured in appropriate units; the origin is taken at the intersection of the meridian with the axis of rotation, which latter coincides with the axis of y. Part I contains tables of the slope tg $\varphi=dy/dx$ and of x as functions of y and h_0. The tables are to five significant figures and cover the range $0.02<y<1.12$ (in steps of 0.02) and $0.004<h_0<0.148$ (in steps of 0.001 for $h_0<0.016$, then in steps of 0.002 for $h_0<0.028$, in steps of 0.004 for $h_0<0.052$, and thence in steps of 0.008). Some additional tables are given for x in terms of y and h_0 and for the equator. Part II deduces from these tables tables for h_0 and for the meniscus volume in terms of x and y. Part III gives analytical approximate solutions for large drops.

The investigation was carried out in view of applications in manometry. The paper contains ample references to already existing tables and known approximate solutions.

W. Feller (Providence, R. I.).

Bödewadt, U. T. Von den freien Schwingungen eines Kreiselpendels bei endlichen Ausschlägen. Z. Angew. Math. Mech. **20**, 218–234 (1940). [MF 3113]

The following three systems of differential equations are considered:

$$\begin{array}{lll}(1) & \ddot\psi+A\sin\psi-N\cos\theta\dot\theta=0, & \ddot\theta+B\sin\theta+M\cos\psi\dot\psi=0;\\ (2) & \ddot\psi+A\psi-N\cos\theta\dot\theta=0, & \ddot\theta+B\sin\theta+M\dot\psi=0;\\ (3) & \ddot\psi+A\psi-N\dot\theta=0, & \ddot\theta+B\theta+M\dot\psi=0;\end{array}$$

where A, B, M, N are positive constants, (2) is an approximation of (1) when ψ is small, (3) is an approximation of (1) and (2) when both ψ and θ are small. The detailed discussion is limited to the system (2), the periodic solutions of which are obtained from the periodic solutions of the linear system (3). The methods employed were originally introduced by Poincaré and elaborated by J. Horn [Z. Math. Phys. **48**, 400–434 (1903)]. Considerable numerical computation is included.

D. C. Lewis (Durham, N. H.).

Crank, J., Hartree, D. R., Ingham, J. and Sloane, R. W. Distribution of potential in cylindrical thermionic valves. Proc. Phys. Soc. **51**, 952–971 (1939). [MF 1511]

By a series of reductions, the physical problem was made to depend on the solution of the following differential equation:

$$\phi''+\frac{4}{3}\phi'+\frac{4}{9}\phi=\frac{4}{9}\phi^{-1/2}.$$

This equation was solved by the differential analyzer for 24 sets of boundary values covering the useful range. Tables

of these solutions are included in the paper. The quantitative work with the differential analyzer was guided by a qualitative study of the first order equation obtained by substituting p for ϕ'. The latter equation was solved graphically by the method of isoclinals. This afforded a means of classifying the solutions of the original equation. Since the differential analyzer is not suited to integration near a singularity of the differential equation, solutions starting with initial condition $\phi=0$ were computed by series or by numerical approximation for a short distance, until a point was reached where machine integration was feasible.

P. W. Ketchum (Urbana, Ill.).

Pekeris, C. L. A pathological case in the numerical solution of integral equations. Proc. Nat. Acad. Sci. U. S. A. 26, 433–437 (1940). [MF 2348]

The author gives examples showing that, in the integral equation

$$f(u)=\int_0^{\infty}(u+v)^{-\frac{1}{2}}H(v)dv,$$

two decidedly different functions $H(v)$ may yield functions $f(u)$ nearly equal (for example, a 25 percent difference in the H's and a 0.1 percent difference in the f's).

W. E. Milne (Corvallis, Ore.).

✱Table of the First Ten Powers of the Integers from 1 to 1000. Published under the sponsorship of the National Bureau of Standards as a report of Official Project No. 365–97–3–11, conducted under the auspices of the United States Works Progress Administration for the City of New York, 1939. viii+80 pp.

The work is in four sections of 20 pages each and gives the nth powers of the integers not greater than 1000 for $n=2, 3, 4, 5$ in Section I; $n=6, 7$ in Section II; $n=8, 9$ in Section III; $n=10$ in Section IV. In each section all the corresponding powers of 50 consecutive integers are given on each page. The elaborate methods employed and precautions taken in computing and checking the stencils from which the table was mimeographed, as described in the introduction, would not have been feasible had not the table been the product of a great many workers. Some irregularities in the mimeographing, though displeasing to the eye of the casual reader, are not serious enough to render uncertain, to the actual user, any of the 10,000 entries of the table. This table was prepared chiefly for use by the project's computing staff in connection with their work on power series and "only a limited number of additional copies were prepared," inasmuch as the British Association for the Advancement of Science is about to publish a more extensive table of powers.

D. H. Lehmer.

✱Tables of Sines and Cosines for Radian Arguments. Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 765–97–3–10; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1940. xix+275 pp. $2.00.

The principle table gives in 250 large pages the values of $\sin x$ and $\cos x$ to eight decimal places at intervals of 0.001 from 0.000 to 25.000 radians. A second table gives values of the same functions with the same accuracy at intervals of unity from 0 to 100 radians. A third table gives these functions to fifteen decimal places at decimal intervals from $1\cdot 10^{-6}$ to $9\cdot 10^{-1}$. A fourth is a conversion table from radians to degrees or to degrees, minutes and seconds, from 0.0001 to 100 radians; and also from degrees, minutes and seconds to radians from 0 seconds to 180 degrees. Table V gives $\sin x$ and $\cos x$ to twelve decimal places at an interval of 0.00001 from 0 to 0.00999. Table VI gives $p(1-p)$ to six decimal places at intervals of 0.001 from 0 to 0.500. The figures are in large type and the tables are easy to read. In an introduction the methods of calculation are described, and also the methods of checking by difference formulas.

E. J. Moulton (Evanston, Ill.).

✱Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments. Prepared by Federal Works Agency, Works Projects Administration for the City of New York, as a Report of Official Project No. 765–97–3–10; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1939. xvii+405 pp. $2.00.

The principal tables are: (1) a table of circular and hyperbolic sines and cosines for arguments from 0.0000 to 1.9999 radians, in steps of 0.0001 radian, to nine decimal places; (2) a table of circular and hyperbolic sines and cosines from 0.0 to 10.0 radians, in steps of 0.1 radian, to nine decimal places. Supplementary tables give conversions from degrees, minutes and seconds, to radians, and from radians to either degrees, minutes, seconds or to degrees and decimal fractions.

Values in the principal tables were computed from the defining infinite series for each function. The work was systematized by computing first the function $U_n(x)=x^n/n!$, for all required values of x and n. Besides computing the desired functions in terms of groupings of $U_n(x)$ terms, a valuable check was obtained by using the same data to compute e^x for comparison with existing tables. All steps in the compilation were checked against independent computations, and the final result was checked by applying differencing tests. The elaborate precautions taken would seem to substantiate the claim of an error of 0.51 unit or less in the ninth decimal place.

The book is reproduced by a photo offset process directly from typewritten sheets. There are a few faint impressions, but the figures are large and legible and the typing is very uniform.

S. H. Caldwell (Cambridge, Mass.).

✱Tables of the Exponential Function e^x. Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 765–97–3–10. Conducted under the Sponsorship of the National Bureau of Standards, 1939. Introduction and 535 pp. $2.00.

In these tables 15 or more place values are given for over 50,000 arguments. Specifically, values are given to 18 decimals from -2.5000 to 1.0000 at intervals of .0001; to 15 decimals from 1.0000 to 2.5000 at intervals of .0001 and from 2.500 to 5.000 by intervals of .001; and to 12 decimals from 5.00 to 10.00 at intervals of .01. The most extensive previous tables are at intervals of .001. The reviewer did not test the accuracy of the tables. The only defect that he noticed was the slightly uneven character of the printing (photo offset process reproduction from typewritten copy). One page may be slightly dim and the next slightly smudged. There are a very few digits which are nearly if not entirely illegible (for instance, the 13th place in the value for -1.5917 and the last place for -1.7530). On the whole the legibility is good, however.

P. W. Ketchum.

AUTHOR INDEX

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